Math 308R: Bridge to Advanced Mathematics

Homework #10

Due date: Tuesday November 29, 2016, 3:30PM

1. Let $f\colon (0,3) \to (1,\infty)$ be the function defined by $f(x) := \frac{3}{x}$.

(a) Prove that f is well-defined.

(b) Prove that f is bijective.

(c) Explain why we can conclude from the previous two items that $|(0,3)| = |(1,\infty)|$.

2. Let $g\colon \{0,1,2,3,4,\dots\} \to \mathbb{Z}$ be the function defined by

$$g(k) := 0 + 1 + \cdots + k.$$

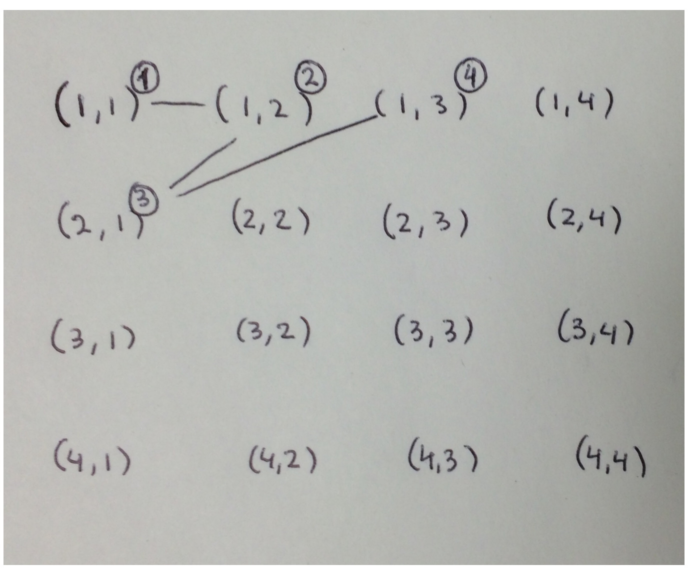
(a) Is the function g bijective?

Let $h\colon \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be the function defined by $h(a,b) := a + g(a+b-2)$.

For example, $h(2,1) = 2 + g(2+1-2) = 2 + g(1) = 2 + 1 = 3$.

(b) In the following 4 by 4 table, calculate the values of h and connect them in order by a line.

(Four values of h have already been filled in, as circled numbers 1 – 4.)



(c) Explain why the function h is bijective.

For problem 3, see other side!

3. Read the following start of a proof; the questions follow the text.

Theorem. Let A be a set. Then $|A| \neq |\mathcal{P}(A)|$.

Proof. We need to prove that there does not exist a bijection $A \to \mathcal{P}(A)$. Reasoning towards a contradiction, suppose that there exists a bijective function $f\colon A \to \mathcal{P}(A)$. Define

$$S := \{a \in A \ : \ a \notin f(a)\}.$$

(a) For example, let $A = \{1, 2, 3\}$, and let $f\colon A \to \mathcal{P}(A)$ be the function defined by $f(1) = \{2, 3\}$, $f(2) = \{2\}$ and $f(3) = \emptyset$. List the elements of the set S.

(b) In this example, does there exist $a \in \{1, 2, 3\}$ such that $f(a) = S$?

We now return to the general proof.

Since f is onto, pick $a \in A$ such that $f(a) = S$.

There are two cases: $a \in S$ or $a \notin S$.

(c) Prove that $a \in S$ gives a contradiction.

(d) Prove that $a \notin S$ also gives a contradiction.