Math 308R: Bridge to Advanced Mathematics

Homework #6

Due date: Tuesday October 18, 2016, 3:30PM

1. For every integer $n \geq 12$, consider the statement

$$P(n): \text{ there exist integers } a, b \geq 0 \text{ such that } n = 4a + 5b.$$

(a) Prove that the statements $P(12)$, $P(13)$, $P(14)$ and $P(15)$ are true.

(b) Prove, using strong induction, that $P(n)$ is true for every $n \geq 12$.

2. If A is a finite set, define

$$\mathcal{P}_{\leq 2}(A) = \{B \subseteq A \ : \ |B| \leq 2\}.$$

Thus, $\mathcal{P}_{\leq 2}(A)$ is the set of subsets of A which have cardinality at most 2.

(a) Give $\mathcal{P}_{\leq 2}(\{x\})$, $\mathcal{P}_{\leq 2}(\{x, y\})$, and $\mathcal{P}_{\leq 2}(\{x, y, z\})$.

(b) Prove that, for any finite set A, if $|A| = n$, then $|\mathcal{P}_{\leq 2}(A)| = 1 + n + \frac{n(n-1)}{2}$.

3. Prove that, for every natural number n: for any integer x, if $3 \mid x^n$, then $3 \mid x$.

4. Consider the sequence $T_1, T_2, T_3, T_4, \ldots$, where

$$T_1 = 1, T_2 = 2, T_3 = 3, \text{ for every } n \geq 4, T_n = T_{n-3} + T_{n-2} + T_{n-1}.$$

(a) Calculate the values T_4, T_5 and T_6.

(b) Prove that, for every integer $k \geq 2$, $2^{k-1} + 2^{k-2} < 2^k$.
(Hint: this can be proved without using induction.)

(c) Prove that, for every natural number n, $T_n < 2^n$.
(Hint: use strong induction, and use item (b) as a Lemma.)