Math 308R: Bridge to Advanced Mathematics

Homework #9

Due date: Tuesday November 22, 2016, 3:30PM

1. Read the following start of a proof; the questions follow the text.

Theorem. Let A and B be finite sets such that $|A| = n = |B|$. Define the set $\mathbf{Bi}(A, B) := \{f : A \to B \ : \ f \text{ is bijective}\}$. Then $|\mathbf{Bi}(A, B)| = n!$.

Proof. We prove, by induction on n, that $\forall n, P(n)$, where $P(n)$ is the statement

$$P(n)\colon \text{for any finite sets } A, B \text{ such that } |A| = n = |B|, |\mathbf{Bi}(A, B)| = n!\,.$$

Base case. If $n = 1$, write $A = \{a\}$ and $B = \{b\}$. Then there is one bijection from A to B, namely $\{(a, b)\}$. So $|\mathbf{Bi}(A, B)| = 1 = 1!$.

Induction step. Suppose $P(k)$ holds. We prove $P(k + 1)$. Let A and B be finite sets such that $|A| = k + 1 = |B|$. Pick an element a of A. For every b in B, define the set

$$\mathbf{Bi}(A, B)_{a\mapsto b} := \{f \in \mathbf{Bi}(A, B) \ : \ f(a) = b\}.$$

(a) For example, let $A = \{1, 2, 3\}$ and $B = \{x, y, z\}$. List the two bijections that are in the set $\mathbf{Bi}(A, B)_{1\mapsto x}$?

(b) In the example from (a), is the collection $\{\mathbf{Bi}(A, B)_{1\mapsto x}, \mathbf{Bi}(A, B)_{1\mapsto y}, \mathbf{Bi}(A, B)_{1\mapsto z}\}$ a partition of $\mathbf{Bi}(A, B)$?

We now return to the general case and finish the proof.

(c) Prove that the collection $\mathcal{S} := \{\mathbf{Bi}(A, B)_{a\mapsto b} \ : \ b \in B\}$ is a partition of $\mathbf{Bi}(A, B)$ into $k + 1$ classes.

(d) Explain why, for every $b \in B$, $|\mathbf{Bi}(A, B)_{a\mapsto b}| = |\mathbf{Bi}(A \setminus \{a\}, B \setminus \{b\})|$.

(e) Using items (c) and (d) and the induction hypothesis, $P(k)$, prove that

$$|\mathbf{Bi}(A, B)| = (k + 1) \cdot k!\,,$$

which is equal to $(k + 1)!$.

2. Let f be the relation from $\{1, 2, 3, 4\}$ to $\mathbb{Q}$ defined by

$$f := \{(1, \frac{5}{6}), (2, \frac{2}{5}), (3, \frac{5}{3}), (4, \frac{2}{5}\}.$$

(a) Prove that f is a function.

(b) Prove or disprove: f is one-to-one.

(c) Prove or disprove: f is onto.

(d) Let $A := \{x \in \mathbb{Q} : 0 \leq x \leq \frac{1}{2}\}$. List the elements of the inverse image set, $f^{-1}(A)$.

3. Prove or disprove, for any functions $f\colon A \to B$ and $g\colon B \to C$,

(a) If $g \circ f$ is injective, then f is injective.

(b) If $g \circ f$ is surjective, then f is surjective.

4. Let $f\colon [0,2] \to [3,7]$ be the function defined by $f(x) := x^2 + 3$.

(a) Prove that f is injective.

(b) Prove that f is surjective.

(c) Define the inverse function $g\colon [3,7] \to [0,2]$ of f.

5. For every natural number n, let $\mathcal{S}_n$ be the set of permutations on $\{1,\dots,n\}$, let $\mathcal{T}_n$ be the set of subsets of $\{1,\dots,n\}$ which do not have cardinality $n-1$, and, for every $\alpha \in \mathcal{S}_n$, let $f_n(\alpha)$ be the set of *fixpoints* of α. In other words,

$$\mathcal{S}_n := \{\alpha \in \{1,\dots,n\}^{\{1,\dots,n\}} : \alpha \text{ is bijective}\},$$

$$\mathcal{T}_n := \{F \in \mathcal{P}(\{1,\dots,n\}) : |F| \neq n-1\},$$
$$f_n(\alpha) := \{k \in \{1,\dots,n\} : \alpha(k) = k\}.$$

(a) Prove that, for every $\alpha \in \mathcal{S}_n$, $f_n(\alpha) \in \mathcal{T}_n$. (Hint. There are two cases: either α is the identity function, or α is different from the identity function.)

(b) Prove that $f_n\colon \mathcal{S}_n \to \mathcal{T}_n$ is onto for every natural number n.

(Hint. You may use as a Lemma that for any finite set S, if $|S| \geq 2$, then there exists a permutation α on S such that, for any $s \in S$, $\alpha(s) \neq s$.)

(c) Prove that $|\mathcal{T}_n| = 2^n - n$ for every natural number n.

(Hint. You may use as a Lemma that $|\mathcal{P}(\{1,\dots,n\})| = 2^n$.)

(d) Use items (b), (c), and the fact that $|\mathcal{S}_n| = n!$ to prove that $n! \geq 2^n - n$ for every natural number n.