Math 308R: Bridge to Advanced Mathematics

Solutions to Midterm Exam 1, 22 September 2016

Note. Below you find one possible solution to each problem; other correct solutions are often possible.

1. Let n be an integer.

(a) Prove: if $5n + 3$ is odd then n is even.

We give a proof by contrapositive. Suppose that n is odd. Then $n = 2x + 1$ for some integer x. Therefore,

$$5n + 3 = 5(2x + 1) + 3 = 10x + 8 = 2(5x + 4),$$

which is even, because $5x + 4$ is an integer.

(b) Prove, using item (a): if $5n + 3$ is odd then $3n - 4$ is even.

Assume that $5n + 3$ is odd. By item (a), n is even. Thus, $n = 2y$ for some integer y. Therefore,

$$3n - 4 = 6y - 4 = 2(3y - 2),$$

which is even, because $3y - 2$ is an integer.

(c) Give the missing word in the following sentence: "When we proved the result in (b), we used the result in (a) as a __________."

Lemma.

2. State the following two statements in words:

(a) $\exists x \in \mathbb{R}, x^2 = -1$.

There exists a real number x such that x^2 is equal to -1.

(b) $3 \mid a \iff a \equiv 2 \pmod 4$.

3 divides a if and only if a is congruent to 2 modulo 4.

State **the negation** of the following two statements in words:

(c) John is driving only if John is wearing a seatbelt.

John is driving and John is not wearing a seatbelt. **or:**
It is not the case that John is driving only if John is wearing a seatbelt.

(d) For every natural number n, if n is odd, then n^2 is odd or $n - 3$ is even.

There exists a natural number n such that n is odd, n^2 is even, and $n - 3$ is odd. **or:**
It is not the case that for every natural number n, if n is odd, then n^2 is odd or $n - 3$ is even.

3. Let A, B, and C be subsets of a universal set U.

(a) For $U = \mathbb{N}$, give an example of three sets A, B and C that are pairwise disjoint.

We need an example of sets of natural numbers A, B, C such that the three sets $A \cap B$, $A \cap C$ and $B \cap C$ are empty. For example, $A = \{1\}$, $B = \{2, 3, 4\}$ and $C = \{5, 6, 7\}$.

(b) Prove that $A - (B \cap C) = (A \cap \overline{B}) \cup (A \cap \overline{C})$. You may use any proof method you wish. If you use any laws, state their names.

$$\begin{aligned} A - (B \cap C) &= A \cap \overline{B \cap C} && \text{(since } A - D = A \cap \overline{D}\text{)} \\ &= A \cap (\overline{B} \cup \overline{C}) && \text{(by de Morgan's law)} \\ &= (A \cap \overline{B}) \cup (A \cap \overline{C}) && \text{(by the distributive law).} \end{aligned}$$

4. Let x and y be real numbers.

(a) Prove that, if $x^3 + y - xy \geq 0$, then $x \geq 0$ or $y \geq 0$.

Assume that $x < 0$ and $y < 0$. Then $x^3 < 0$ and $xy > 0$, so $-xy < 0$. Therefore, $x^3 + y - xy < 0$.

(b) Give the name of the proof method you used in item (a).

Proof by contrapositive.

5. For each $k \in \{0, 1, 2, 3\}$, let A_k be the set $\{x \in \mathbb{Z} \mid x \equiv k \pmod 4\}$.

(a) Is $\{A_0, A_1, A_2, A_3\}$ a partition of $\mathbb{Z}$? You should state the definition of 'partition' to explain your answer.

Yes. It satisfies the three conditions for being a partition, namely, the sets A_k are pairwise disjoint ($A_k \cap A_\ell = \emptyset$ when $k \neq \ell$), their union is $\mathbb{Z}$ ($\bigcup_{k=0}^{3} A_k = \mathbb{Z}$), and none of the sets is empty ($A_k \neq \emptyset$ for each k).

(b) Prove that, for every integer y: $y^2 \in A_1$ if and only if $y \in A_1$ or $y \in A_3$. You may use the following fact without proving it: for any integers x and y, if $y \equiv x \pmod 4$, then $y^2 \equiv x^2 \pmod 4$.

Let y be an integer. First assume that $y \in A_1$ or $y \in A_3$.

Case 1. $y \in A_1$. By definition, $y \equiv 1 \pmod 4$. By the fact, $y^2 \equiv 1 \pmod 4$, so $y^2 \in A_1$.

Case 2. $y \in A_3$. By definition, $y \equiv 3 \pmod 4$. By the fact, $y^2 \equiv 9 \pmod 4$. Since $9 \equiv 1 \pmod 4$, $y^2 \in A_1$.

Conversely, assume that $y \notin A_1$ and $y \notin A_3$. There are two cases: $y \in A_0$ or $y \in A_2$.

Case 1. $y \in A_0$. Then $y \equiv 0 \pmod 4$, so $y^2 \equiv 0 \pmod 4$ by the fact. So $y^2 \in A_0$. Hence, $y^2 \notin A_1$.

Case 2. $y \in A_2$. Then $y \equiv 2 \pmod 4$, so $y^2 \equiv 4 \pmod 4$ by the fact. Since $4 \equiv 0 \pmod 4$, it follows that $y^2 \in A_0$. Hence, $y^2 \notin A_1$.

6. Let P, Q and R be statements.

(a) Prove, using a truth table, that $P \Rightarrow (Q \Rightarrow P)$ is a tautology.

P	Q	$Q \Rightarrow P$	$P \Rightarrow (Q \Rightarrow P)$
T	*T*	*T*	*T*
T	*F*	*T*	*T*
F	*T*	*F*	*T*
F	*F*	*T*	*T*

Since the truth values in the column $P \Rightarrow (Q \Rightarrow P)$ are all T, it is a tautology.

(b) Prove, without using a truth table, that $P \Rightarrow {\sim}(Q \wedge R) \equiv {\sim}P \vee {\sim}Q \vee {\sim}R$. (Hint: use the fact that, for any statements P and S, $P \Rightarrow S \equiv {\sim}P \vee S$, and use de Morgan's laws. Clearly state when you use these facts.)

$$\begin{aligned} P \Rightarrow {\sim}(Q \wedge R) &\equiv {\sim}P \vee {\sim}(Q \wedge R) && \text{(using } P \Rightarrow S \equiv {\sim}P \vee S\text{)} \\ &\equiv {\sim}P \vee {\sim}Q \vee {\sim}R. && \text{(using de Morgan's law)} \end{aligned}$$