Math 308R: Bridge to Advanced Mathematics

Solutions to Midterm Exam 2, 8 November 2016

Note. Below you find one possible solution to each problem; other correct solutions are often possible.

1. *This problem is about $\mathbb{Z}_6$.*

(a) *Give the definition of the set $\mathbb{Z}_6$.*

The set $\mathbb{Z}_6$ is the set of integers modulo 6. An integer modulo 6 is an equivalence class of the relation R_6. The relation R_6 is defined by $aR_6b \iff 6$ divides $b - a$.

(b) *Prove that multiplication is well-defined in $\mathbb{Z}_6$.*

Suppose that $a \equiv a' \pmod 6$ and $b \equiv b' \pmod 6$. We need to prove that $ab \equiv a'b' \pmod 6$. Pick integers x and y such that $a' - a = 6x$ and $b' - b = 6y$. Then

$$a'b' - ab = (a' - a)b' + (b' - b)a = 6xb' + 6ya = 6(xb' + ya).$$

Hence, $6 \mid a'b' - ab$.

(c) *Prove that, for any integer x, if $6 \mid x - 5$, then $6 \mid x^3 - 5$.*

Suppose that $6 \mid x - 5$. This means that $[x] = [5]$ in $\mathbb{Z}_6$. Therefore, since multiplication is well-defined on $\mathbb{Z}_6$, $[x^3] = [5^3] = [125] = [5]$ in $\mathbb{Z}$. So $6 \mid x^3 - 5$.

2. *Prove or disprove:*

(a) *For any sets A, B, and C, $A - (B \cap C) \subseteq (A - B) \cup (A - C)$.*

We prove the statement. Let $a \in A - (B \cap C)$ be arbitrary. Then $a \in A$ and $a \notin B \cap C$. Therefore, $a \notin B$ or $a \notin C$. Without loss of generality, assume $a \notin B$. Then $a \in A - B$. Hence, $a \in (A - B) \cup (A - C)$.

(b) *There exists an integer x such that x is odd and $x \equiv 2 \pmod 8$.*

We disprove the statement. We need to prove that, for every integer x such that x is odd, $x \not\equiv 2 \pmod 8$. Let x be an odd integer. If we would have $x \equiv 2 \pmod 8$, then there would exist an integer y such that $x - 2 = 8y$. But then $x = 8y + 2 = 2(4y + 1))$, so x is even, which is a contradiction with the assumption that x is odd. Therefore, $x \not\equiv 2 \pmod 8$.

3. *Prove that, for every irrational number x and rational number y such that $y \neq 0$, the product $x \cdot y$ is irrational. (Hint: Use a proof by contradiction.)*

Let x be an irrational number and y a rational number such that $y \neq 0$. Choose non-zero integers p, q such that $y = \frac{p}{q}$. Reasoning towards a contradiction, suppose that $x \cdot y$ is rational. Choose integers r and $s \neq 0$ such that $x \cdot y = \frac{r}{s}$. Then

$$x = \frac{x \cdot y}{y} = \frac{rq}{ps},$$

which is rational, contradicting the assumption that x is irrational.

4. *Prove: For every integer $n \geq 12$, there exist nonnegative integers a and b such that $n = 3a + 7b$.*
We prove by strong induction that $\forall n \geq 12, P(n)$, where $P(n)$ is the statement

There exist nonnegative integers a and b such that $n = 3a + 7b$.

Base cases. Since $12 = 3 \cdot 4 + 7 \cdot 0$, $13 = 3 \cdot 2 + 7 \cdot 1$, and $14 = 3 \cdot 0 + 7 \cdot 2$, the statements $P(12)$, $P(13)$ and $P(14)$ are true.
Strong induction step. Suppose $P(i)$ holds for all $12 \leq i \leq k$, where $k \geq 14$. We prove $P(k+1)$. Since $k - 2 \geq 12$, by $P(k-2)$, pick nonnegative integers a and b such that $k - 2 = 3a + 7b$. Then

$$k + 1 = (k - 2) + 3 = 3(a + 1) + 7b,$$

so $P(k + 1)$ is true.

5. We denote by $\mathbb{Q}_{>0}$ the set of positive rational numbers. Let R be the relation on the set $\mathbb{Q}_{>0}$ defined by

$$xRy \iff \text{there exists an integer } k \text{ such that } x = y \cdot 2^k.$$

(a) *Prove that R is an equivalence relation.*

We prove that R is reflexive, symmetric and transitive.

- R is reflexive. For any $x \in \mathbb{Q}_{>0}$, $x = x \cdot 2^0$, so xRx.
- R is symmetric. Suppose that xRy. Pick an integer k such that $x = y \cdot 2^k$. Then $y = x \cdot 2^{-k}$. Hence, yRx.
- R is transitive. Suppose that xRy and yRz. Pick integers k and ℓ such that $x = y \cdot 2^k$ and $y = z \cdot 2^\ell$. Then

$$x = y \cdot 2^k = z \cdot 2^\ell \cdot 2^k = z \cdot 2^{\ell+k}.$$

Hence, xRz.

(b) *Prove that $[\frac{5}{2}] = [40]$.*

(Hint: You may use, as a Lemma, that for any x, y, xRy if, and only if, $[x] = [y]$. You don't need to prove this Lemma.)

Since $\frac{5}{2} = \frac{40}{16} = 40 \cdot 2^{-4}$, we have $(\frac{5}{2}, 40) \in R$. By the Lemma, $[\frac{5}{2}] = [40]$.

(c) *Give three elements of $[\frac{1}{3}]$.*

By definition, $[\frac{1}{3}] = \{b \in \mathbb{Q}_{>0} : \frac{1}{3} = b \cdot 2^k \text{ for some integer } k\}$. For example, $\frac{1}{3}$, $\frac{2}{3}$ and $\frac{4}{3}$ are elements of $[\frac{1}{3}]$.

6. *Consider the following statement $P(n)$:*

For every integer x, $x \equiv 0 \pmod{n}$ or there exists an integer y such that $xy \equiv 1 \pmod{n}$.

(a) *State the negation of the statement* $P(n)$.

There exists an integer x such that $x \not\equiv 0 \pmod{n}$, and, for every integer y, $xy \not\equiv 1 \pmod{n}$.

(b) *Disprove the statement* $P(10)$.

We prove the negation of $P(10)$. For example, let $x := 2$. Let y be any integer. Since xy is even, $xy - 1$ is odd. Therefore, 10 does not divide $xy - 1$. This means that $xy \not\equiv 1 \pmod{10}$.

(c) *Prove that there exists a natural number* $n \geq 2$ *such that* $P(n)$ *is true.*

We need to give an example of a natural number $n \geq 2$ such that $P(n)$ is true. For example, take $n = 2$ (any prime number will work). We prove that $P(2)$ is true.

Let x be an integer. There are two cases:

- x is even. Then $x \equiv 0 \pmod{2}$, so we are done.
- x is odd. Let $y := 1$. Then $xy \equiv 1 \pmod{2}$, since $xy = x$.