Logic, Automata, and Model Companions

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The aim of this talk is to show a connection between temporal logics and monadic second order (MSO) logics on discrete structures, mediated by model theory.
In formal language theory, logic can be used as a descriptive formalism which measures the complexity of a computational problem. For example, the MSO-definable sets of finite words are exactly the regular languages from automata theory. Büchi and Rabin established such translations between MSO logic and automata on many more structures, including omega-indexed words and various types of trees.
In model theory, logic can be used to give a general account of a mathematical construction; particularly relevant to this talk is the construction of the algebraic closure of a field. A fundamental insight due to Robinson is that the notion of algebraically closed field can be generalized to a purely logical notion of "existentially closed model". The "model companion" of a first order theory, if it exists, gives a first order description of the class of existentially closed models.
This talk's main thesis will be that MSO logic 'is' the model companion of temporal logic. That is, monadic second order logic is to temporal logic as algebraically closed fields are to fields. Indeed, in joint work with Ghilardi, we proved such a model companion result both for words [1] and for trees up to bisimulation [2]. Extending these results to full MSO on trees is the subject of ongoing work. In the remainder of this abstract, I will make the statement of the result for words [1] more precise.

Monadic second order logic on ω, also known as S1S (second order logic of one successor), is defined by adding to the first-order logic of the successor function quantification over unary predicates, i.e., subsets of ω. S1S can be used to define sets of *streams* over a finite alphabet Σ, i.e., functions $S\colon \omega \to \Sigma$. Let φ be a formula of S1S, all of whose free second-order variables are in a finite set V. Then valuations $v\colon V \to \mathcal{P}(\omega)$ are in one-to-one correspondence with $\mathcal{P}(V)$-streams $S_v\colon \omega \to \mathcal{P}(V)$, and the formula φ thus defines a set of $\mathcal{P}(V)$-streams:

$$L_\varphi := \{S_v \;:\; \omega, v \models \varphi\}.$$

A different way of arriving at languages of Σ-streams, where now Σ is any finite alphabet, is by using non-deterministic finite automata (NFA). A language of Σ-streams is *ω-regular* if it is recognized by some NFA. Here, the definitions of automata are the same as for finite words, except that the acceptance condition now says that there is a run which visits a final state infinitely often. Büchi proved that the stream languages of the form L_φ are exactly the ω-regular languages. In fact, he gave an effective procedure which transforms an S1S formula into an automaton, and vice versa. The S1S formula which is associated to an automaton in this procedure has a special form: all the second-order quantifications are *existential*. From

Büchi's result, one may thus obtain a 'normal form' for S1S, by translating into an automaton and back.

This normal form result is what allows us to prove that a certain first-order theory, T^*, closely related to S1S, is model complete. Here, a first-order theory T^* is called *model complete* if for every formula φ, there is an existential formula φ' such that $T^* \vdash \varphi \leftrightarrow \varphi'$. Thus, a model complete theory 'almost' has quantifier elimination, up to the last layer of quantifiers. The *model companion* of a universal first-order theory T is the model complete theory T^* which has the same universal consequences as T. The model companion of T is unique if it exists, in which case it is the first order theory of the class of existentially closed models for T.

Finally, what is the temporal logic that S1S is a model companion of? Denote by $\mathcal{L}$ the one-way, discrete, linear temporal logic with unary operators 'next' and 'future', and a constant 'initial moment'. That is, the syntax of $\mathcal{L}$ is the propositional language is enriched with unary symbols $\mathbf{X}$ and $\mathbf{F}$, and a nullary symbol $\mathbf{I}$. The algebraic models for $\mathcal{L}$ are Boolean algebras with operators in this signature, subject to axioms expressing that (i) $\mathbf{X}$ is a Boolean endomorphism; (ii) $\mathbf{F}a$ is the least fixpoint of the function $x \mapsto a \vee \mathbf{X}x$; (iii) $\mathbf{I}$ is an atom such that $\mathbf{XI} = \bot$, and $\mathbf{I} \leq \mathbf{F}a$ whenever $a \neq \bot$. Write TA for the universal first order theory axiomatizing this class of temporal algebras. The prototypical example of a temporal algebra is the Boolean algebra $\mathcal{P}(\omega)$, equipped with temporal operators $\mathbf{X}a := \{x \in \omega \mid x + 1 \in a\}$, $\mathbf{F}a := \{x \in \omega \mid \exists y \geq x, y \in a\}$ and $\mathbf{I} := \{0\}$. Write TA^* for the first order theory of this particular temporal algebra. Clearly, looking at first order formulas in the algebra $\mathcal{P}(\omega)$ is almost the same thing as looking at monadic second order formulas interpreted in ω, thus, $\mathsf{TA}^* \approx S1S$. Our main result in [1] is:

Theorem. The theory TA^* is the model companion of the theory TA.

The proof of this result involves two parts: the first is the normal form procedure mentioned above, the second is a completeness result for the logic $\mathcal{L}$ with respect to the intended model ω, for which we give a short proof based on a Stone-Jónsson-Tarski style duality for temporal algebras.

The above theorem establishes that S1S, when viewed as the first-order theory of the temporal algebra $\mathcal{P}(\omega)$, 'is' the model companion of the linear temporal logic $\mathcal{L}$ described above. In [2], we extend this result to monadic second order logic S2S of two successors, which is interpreted on binary trees, and we also treat arbitrarily branching trees, but there we restrict MSO to its bisimulation-invariant fragment. The temporal logics involved in [2] are more involved: we had to design an extension of computation tree logic (CTL) with binary 'fairness' operators, and prove a completeness result for that. This led us into a complex but interesting study of the completeness of certain fragments of the μ-calculus. In current work in progress, we plan to extend this work to a *graded* temporal logic, towards obtaining a model companion result for full MSO on trees.

References

[1] S. Ghilardi and S. J. van Gool, *A model-theoretic characterization of monadic second-order logic on infinite words*, Journal of Symbolic Logic, vol. 82, no. 1, 62-76 (2017).

[2] S. Ghilardi and S. J. van Gool, *Monadic second order logic as the model companion of temporal logic*, Proc. LICS 2016, 417-426 (2016).