Equations and logic on words

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Overview

Logic on words

Equations between languages

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Regular languages: example

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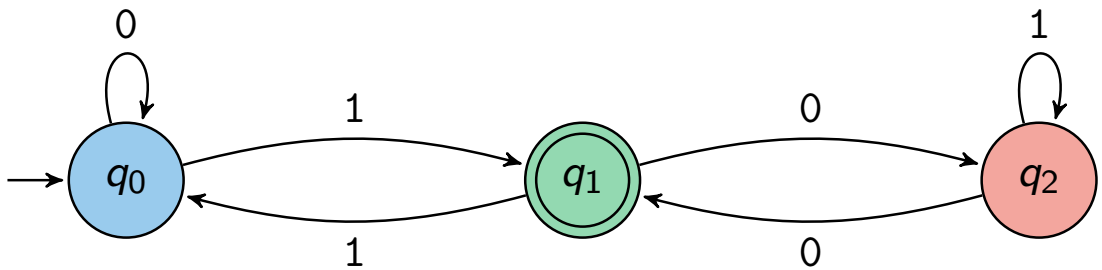

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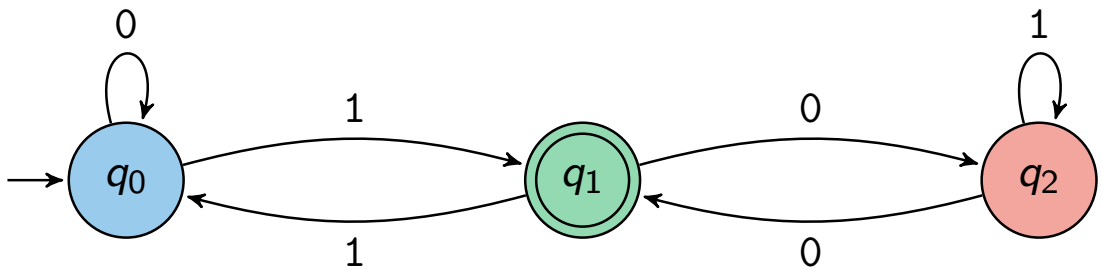

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- Solution 2: a homomorphism $\varphi\colon \{0,1\}^* \to S_3$ defined by

$$0 \mapsto (1\,2), \quad 1 \mapsto (0\,1).$$

  Answer yes iff the permutation $\varphi(w)$ sends 0 to 1.

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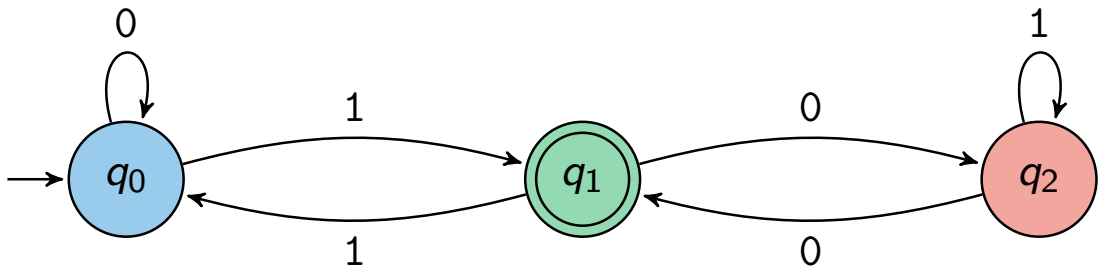

  Answer yes iff A accepts w.

- Solution 3: an MSO sentence φ:

  $\exists Q_0 \exists Q_1 \exists Q_2 (Q_0(\texttt{first}) \wedge Q_1(\texttt{last}) \wedge$

  $\forall x[0(x) \wedge Q_0(x) \rightarrow Q_0(\mathrm{S}x)] \wedge [1(x) \wedge Q_0(x) \rightarrow Q_1(\mathrm{S}x)] \wedge \ldots).$

  Answer yes iff w satisfies the formula φ.

Regular languages

Regular languages are subsets $L \subseteq \Sigma^*$ which are ...

- recognizable by a finite automaton;
- invariant under a finite index monoid congruence;
- definable by a monadic second order sentence.

Myhill-Nerode 1958; Büchi 1960

Logic on words

- **Syntax.** Monadic Second Order (MSO) logic over $<$, Σ.
  - Basic propositional connectives: $\wedge$, $\neg$.
  - Quantification over first-order variables x, y, ... and monadic second-order variables P, Q, ...
  - Relational signature: $x < y$, $\mathrm{a}(x)$ for $a \in \Sigma$.

Logic on words

- **Syntax.** Monadic Second Order (MSO) logic over $<$, Σ.
- **Semantics.** A word $w = a_1 \dots a_n$ gives a structure W.
  - The underlying set of W is $\{1, \dots, n\}$.
  - The natural linear order $<^W$ interprets the binary predicate $<$.
  - For every letter $a \in \Sigma$, $\mathrm{a}^W := \{i \in \{1, \dots, n\} : a_i = a\}$.

Logic on words

- **Syntax.** Monadic Second Order (MSO) logic over $<$, Σ.
- **Semantics.** A word $w = a_1 \dots a_n$ gives a structure W.

- For a sentence φ, $L_\varphi := \{w \in \Sigma^* \mid w \models \varphi\}$.
- A language L is regular iff $L = L_\varphi$ for some φ in MSO.

- Shortcuts such as $\mathtt{S}(x)$, $\mathtt{first}$, $\mathtt{last}$, $\subseteq$, ... are MSO-definable.
- First Order (FO) logic is obtained by disallowing second order variables and second order quantifiers.

Logic on words: examples

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$$\varphi : \exists P \Big[\ P(\texttt{first}) \wedge \neg P(\texttt{last}) \wedge \forall x (P(x) \leftrightarrow \neg P(\texttt{S}(x)))\ \Big].$$

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$\psi\colon \exists P\big[\ \exists x P(x)\ \wedge\ P \subseteq \texttt{a} \wedge \forall y\big(\ (\forall x[P(x) \to x < y])\ \to \texttt{b}(y)\big)\ \big].$

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Question. Does such an equivalent first order formula exist for φ?

Monoids and finite index congruences

- A monoid is a set M equipped with an associative binary operation and a unit.
- The set Σ^* of finite words is a free monoid.
  - multiplication is concatenation;
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- A monoid is a set M equipped with an associative binary operation and a unit.
- The set Σ^* of finite words is a free monoid.
  - multiplication is concatenation;
  - unit is the empty word ϵ;
- A congruence on M is an equivalence relation θ which respects multiplication.
  - The quotient M/θ is again a monoid;
  - A congruence θ is called finite index if M/θ is finite.
- A language $L \subseteq \Sigma^*$ is regular iff there exists finite index θ_L under which L is invariant:

$$w \in L \text{ and } w\theta_L w' \text{ implies } w' \in L.$$

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Duality

Key insight. The connection between MSO logic on words and monoids is an instance of Stone-Jónsson-Tarski duality.

Algebra	**Space**
Lindenbaum algebra of a logic	Canonical model
Residuated Boolean algebra of regular languages	(Pro)finite monoid

Gehrke, Grigorieff, Pin 2008

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Equations between languages	Equations between words

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Equations between words

Solving equations

- Solve for $x \in \mathbb{R}$: $x^2 + 1 = 0$.

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- A structure A is existentially closed* if any existential sentence that becomes true in some extension of A already holds in A.
- This property is often first order definable:
  - Linear orders without endpoints: density;
  - Boolean algebras: atomless;
  - Heyting algebras: mimick fields, use uniform interpolation.

* If the class of structures does not have amalgamation, a more complicated definition is needed.

Model companion

A first order theory T^* which captures the existentially closed models for a universal theory T is called a model companion of T.

Theorem.

The theory T^*, if it exists, is the unique theory such that:

1. T and T^* believe the same universal sentences;

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Robinson, 1963

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   T and T^* are co-theories

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   T^* is model complete

Robinson, 1963

Model companions and languages

Theorem.

The first order theory T^* of an algebra for word languages, $\mathcal{P}(\omega)$,

is the model companion of

a theory T of algebras for a linear temporal logic.

Ghilardi & G. JSL 2017

Model companions and languages

Theorem.

The first order theory T^* of an algebra for tree languages, $\mathcal{P}(2^*)$,

is the model companion of

a theory T of algebras for a fair computation tree logic.

Ghilardi & G. LICS 2016

Proof idea: set-up

Skip

- Enrich the Boolean algebra $\mathcal{P}(\omega)$ with temporal operators:
  - $\mathbf{X}a := \{t \in \omega \mid t+1 \in a\}$,
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- Axioms for temporal logic $\to$ a first order theory T.

Theorem. The theory T^* of $\mathcal{P}(\omega)$ is the model companion of T.

i.e., T^* is model complete and T^* is a co-theory of T.

Proof idea: model completeness

- Any first order formula $\varphi(\overline{p})$ in this temporal algebra translates to an MSO formula $\Phi(\overline{P})$ in logic on words.

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- Describe the automaton A with an existential first order formula φ' in the temporal algebra $\mathcal{P}(\omega)$.

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- This MSO formula Φ defines a regular language L_φ.
- Build an automaton A for Φ.
- Describe the automaton A with an existential first order formula φ' in the temporal algebra $\mathcal{P}(\omega)$.
- **Conclusion.** $\mathcal{P}(\omega)$ believes that any first order formula φ is equivalent to an existential formula φ'.

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Logic on words: example

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Logic and monoids

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if, and only if,

L is invariant under a finite index monoid congruence.

Logic and monoids

A language $L \subseteq \Sigma^*$ is FO-definable

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Logic and monoids

A language $L \subseteq \Sigma^*$ is FO-definable

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L is invariant under a finite index aperiodic monoid congruence.

A congruence θ on Σ^* is called aperiodic if Σ^*/θ does not have non-trivial subgroups.

Schützenberger 1965; McNaughton, Papert 1971

ω

In a finite monoid, any element x has a unique idempotent, x^ω, in its orbit $\{x, x^2, x^3, \dots\}$.

A finite monoid is aperiodic iff it satisfies the equation

$$x^\omega = x^\omega x.$$

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In a profinite monoid, any element x has a unique idempotent, x^ω, in its orbit-closure $\overline{\{x, x^2, x^3, \dots\}}$.

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Decision problem. Given two terms in $\cdot$ and $()^\omega$, are they equal in every finite aperiodic monoid?

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Decision problem. Given two terms in $\cdot$ and $()^\omega$, are they equal in the free profinite aperiodic monoid?

The free profinite aperiodic monoid

Theorem.

The free profinite aperiodic monoid

=

The topological monoid of ultrafilters of FO-definable languages

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The topological monoid of $\equiv_{FO}$-classes of pseudo-finite words.

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Using a concrete interpretation of $\cdot$ and $()^\omega$ via saturated models, the decision problem reduces to checking isomorphism of specific pseudo-finite words.

G. & Steinberg STACS 2017

The future

- Language equations and unification
- Word equations and unification
- From FO to MSO