# Uniform Interpolation and Compact Congruences

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The following remarkable feature of intuitionistic propositional logic (**IPC**) was established by A. M. Pitts in [3]. Given any formula $\alpha(\bar{x}, \bar{y})$ (using brackets as usual to indicate variables that may occur in the formula), there exist formulas $\alpha^L(\bar{y})$ and $\alpha^R(\bar{y})$, *left* and *right uniform interpolants* of $\alpha$, respectively, such that for any formula $\beta(\bar{y}, \bar{z})$,

$$\beta \vdash_{\mathbf{IPC}} \alpha \quad \Leftrightarrow \quad \beta \vdash_{\mathbf{IPC}} \alpha^L \qquad \text{and} \qquad \alpha \vdash_{\mathbf{IPC}} \beta \quad \Leftrightarrow \quad \alpha^R \vdash_{\mathbf{IPC}} \beta.$$

All seven intermediate logics admitting Craig interpolation also admit uniform interpolation; however, although the modal logic **K** admits both properties, its extension **S4** admits only Craig interpolation and not uniform interpolation (see [1] for details and references).

Uniform interpolation for a logic may be viewed as a weaker form of quantifier elimination. This idea is exploited in the monograph [1] of Ghilardi and Zawadowski to show that under certain conditions, satisfied in particular by **IPC** and **K**, uniform interpolation for a logic implies the existence of a model completion for a corresponding variety (equational class) of algebras.

In this work, we investigate uniform interpolation in a universal algebraic setting. Following the category-theoretic work in [1], we obtain algebraic characterizations of the property of existence of left and right uniform interpolants. Moreover, we identify, among varieties of algebras corresponding to substructural and many-valued logics, several varieties that admit and do not admit these properties.

In the remainder of this abstract, we give a more technical description of our main results. Let us fix an algebraic language $\mathcal{L}$ and a variety $\mathcal{V}$ of $\mathcal{L}$-algebras. We denote by $\mathbf{F}_{\mathcal{V}}(\overline{x})$ the free $\mathcal{V}$-algebra over a set of variables $\overline{x}$. The *deductive interpolation property* [2] for $\mathcal{V}$ is easily shown to be equivalent to: for any set of equations $\Sigma(\overline{x}, \overline{y})$, there exists a set of equations $\Pi(\overline{y})$ such that for any equation $\varepsilon(\overline{y}, \overline{z})$, $\Sigma \models_{\mathcal{V}} \varepsilon$ iff $\Pi \models_{\mathcal{V}} \varepsilon$. We now formulate a uniform version of this property: $\mathcal{V}$ has *right uniform deductive interpolation* if, for any finite $\overline{x}$, $\overline{y}$ and any *finite* set of equations $\Sigma(\overline{x}, \overline{y})$, there exists a *finite* set of equations $\Pi(\overline{y})$ such that for any equation $\varepsilon(\overline{y}, \overline{z})$, $\Sigma \models_{\mathcal{V}} \varepsilon$ iff $\Pi \models_{\mathcal{V}} \varepsilon$.

In Theorem 1 below, we translate the above definition into a property of free finitely generated algebras of $\mathcal{V}$. To this end, note first that any homomorphism $f\colon \mathbf{A} \to \mathbf{B}$ lifts to an adjunction $f^*\colon \mathrm{Con}(\mathbf{A}) \leftrightarrows \mathrm{Con}(\mathbf{B})\colon f^{-1}$ between the congruence lattices, with $f^*$ (direct image) left adjoint to $f^{-1}$ (inverse image). Moreover, the map $f^*$ restricts correctly to the sub-join-semilattices of compact (i.e., finitely generated) congruences, $\mathrm{KCon}(\mathbf{A})$ and $\mathrm{KCon}(\mathbf{B})$. We call the restriction of $f^*$ to compact congruences the *compact lift of $f$*. By general lattice-theoretic considerations, the compact lift $f^*\colon \mathrm{KCon}(\mathbf{A}) \to \mathrm{KCon}(\mathbf{B})$ has a right adjoint if, and only if, $f^{-1}$ preserves compact congruences. In this case, the restriction of $f^{-1}$ to compact congruences is that right adjoint. As a first characterization of right uniform deductive interpolation, we have the following.

**Theorem 1.** *For any variety $\mathcal{V}$, the following are equivalent:*

1. *$\mathcal{V}$ has right uniform deductive interpolation;*
2. *(a) for any finite $\overline{x}$, $\overline{y}$, the compact lift of $\mathbf{F}_{\mathcal{V}}(\overline{x}) \hookrightarrow \mathbf{F}_{\mathcal{V}}(\overline{x}, \overline{y})$ has a right adjoint, and*
   *(b) $\mathcal{V}$ has deductive interpolation.*
3. *for any $\overline{x}$, $\overline{y}$, the compact lift of $\mathbf{F}_{\mathcal{V}}(\overline{x}) \hookrightarrow \mathbf{F}_{\mathcal{V}}(\overline{x}, \overline{y})$ has a right adjoint.*

**Examples.** Heyting algebras have right uniform deductive interpolation by [3] and the fact that any Heyting algebra $\mathbf{A}$ is dually isomorphic to $\mathrm{KCon}(\mathbf{A})$. Note that (2a) in Theorem 1 is automatically true in any variety for which any congruence on a finitely generated free algebra is compact.

In particular, any locally finite variety $\mathcal{V}$ with deductive interpolation has right uniform deductive interpolation. Moreover, abelian groups, abelian $\ell$-groups and MV-algebras all have right uniform deductive interpolation. On the other hand, in the variety of algebras for the modal logic S4, (2a) does not hold [1], and (2a) also fails in the variety of groups.[4]

In the next theorem, we show that property (2a) in Theorem 1 guarantees the existence of right adjoints for compact lifts of arbitrary homomorphisms between finitely presented algebras.

**Theorem 2.** *For any variety $\mathcal{V}$, the following are equivalent:*

1. *for any finite $\overline{x}$, $\overline{y}$, the compact lift of $\mathbf{F}_{\mathcal{V}}(\overline{x}) \hookrightarrow \mathbf{F}_{\mathcal{V}}(\overline{x}, \overline{y})$ has a right adjoint;*
2. *for any homomorphism $f\colon \mathbf{A} \to \mathbf{B}$ between finitely presented algebras of $\mathcal{V}$, the compact lift of $f$ has a right adjoint.*

To prove this theorem, we show that one may choose appropriate presentations of $\mathbf{A}$ and $\mathbf{B}$ so that the right adjoint for the compact lift of $f$ can be constructed from the right adjoints that are assumed to exist in (1).

We say that $\mathcal{V}$ has *left uniform deductive interpolation* if, for any finite set of equations $\Delta(\overline{y}, \overline{z})$, there exists a finite set of equations $\Pi(\overline{y})$ such that for any set of equations $\Sigma(\overline{x}, \overline{y})$, $\Sigma \models_{\mathcal{V}} \Delta$ iff $\Sigma \models_{\mathcal{V}} \Pi$. Theorem 1 holds if one replaces 'right' by 'left' throughout. However, the property of left uniform interpolation is not entirely analogous to that of right uniform interpolation.

**Examples.** As above, Heyting algebras have left uniform deductive interpolation by [3]. It follows from the 'left' version of Theorem 1 that a locally finite variety $\mathcal{V}$ has left uniform deductive interpolation if, and only if, $\mathcal{V}$ has deductive interpolation *and* the compact lift of $\mathbf{F}_{\mathcal{V}}(\overline{x}) \hookrightarrow \mathbf{F}_{\mathcal{V}}(\overline{x}, \overline{y})$ preserves intersections. In particular, we use these observations to give an algebraic proof that the variety of Brouwerian meet-semilattices does not have left uniform deductive interpolation.

One may now naturally wonder if an analogous result to Theorem 2 holds for left adjoints. It turns out that an additional condition is needed. We call a join-semilattice *dually Brouwerian* if the operation of binary join has a left residual.

**Theorem 3.** *For any variety $\mathcal{V}$, the following are equivalent:*

1. *for any finite $\overline{x}$, $\overline{y}$, the compact lift of $\mathbf{F}_{\mathcal{V}}(\overline{x}) \hookrightarrow \mathbf{F}_{\mathcal{V}}(\overline{x}, \overline{y})$ has a left adjoint, and $\mathrm{KCon}(\mathbf{F}_{\mathcal{V}}(\overline{x}))$ is a dually Brouwerian join-semilattice;*
2. *for any homomorphism $f\colon \mathbf{A} \to \mathbf{B}$ between finitely presented algebras of $\mathcal{V}$, the compact lift of $f$ has a left adjoint.*

For the proof of this theorem, we first observe that, for any algebra $\mathbf{A}$, the join-semilattice $\mathrm{KCon}(\mathbf{A})$ is dually Brouwerian if, and only if, the compact lift of any surjective homomorphism $p\colon \mathbf{A} \twoheadrightarrow \mathbf{B}$ has a left adjoint. This characterization is subsequently combined with an argument similar to that in the proof of Theorem 2.

## References

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