Finite Automata and Monadic Second Order Logic

Sam van Gool

Monday, April 29, 2019

Logic and Computation, Master's course, Utrecht University

Central question in theory of computation

What are the fundamental capabilities

and limitations of computers?

Three types of sub-questions, plus a bonus question

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  - How *hard* is a computational problem?
  - How much *time*? How much *memory space*?

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  - What is a *minimal* model for computation?
- For each of the above, what is the connection to *logic*?

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- **Reminder!** Top-3 topic preferences due on Blackboard on Wednesday at 5pm.
- **Reminder!** Research question & choice of peer reviewer on Friday at 5pm.

Overview for today

Deterministic Finite Automata

Regular languages and non-determinism

Monadic Second Order Logic

The NFA-MSO Connection

Deterministic Finite Automata

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- ‘New Computer’ (external video link)

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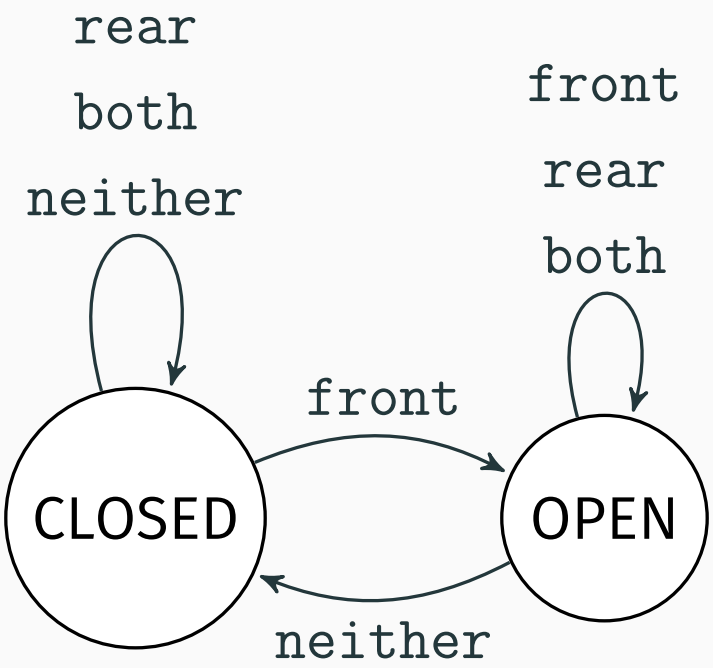
Automatic door automaton



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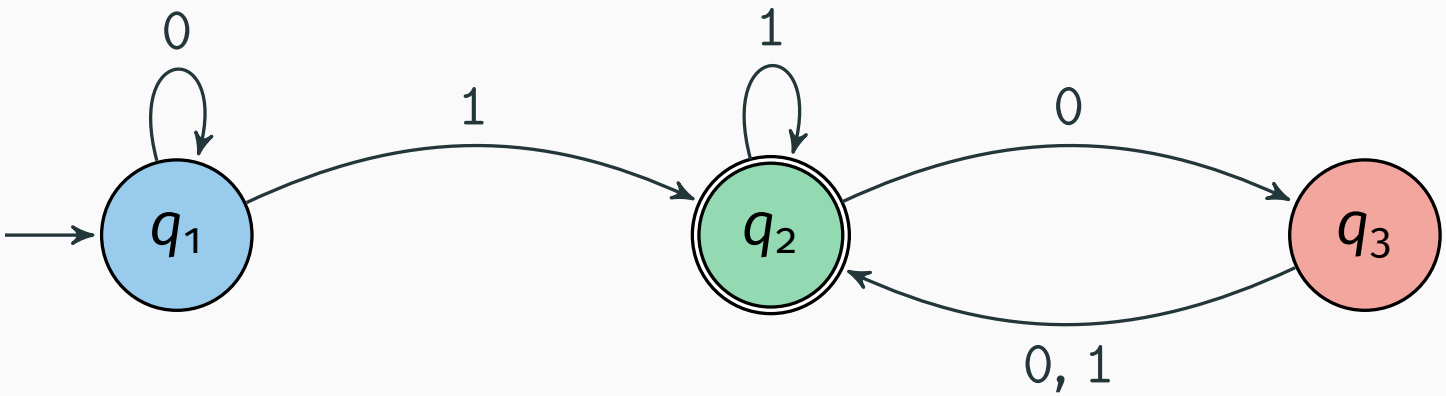

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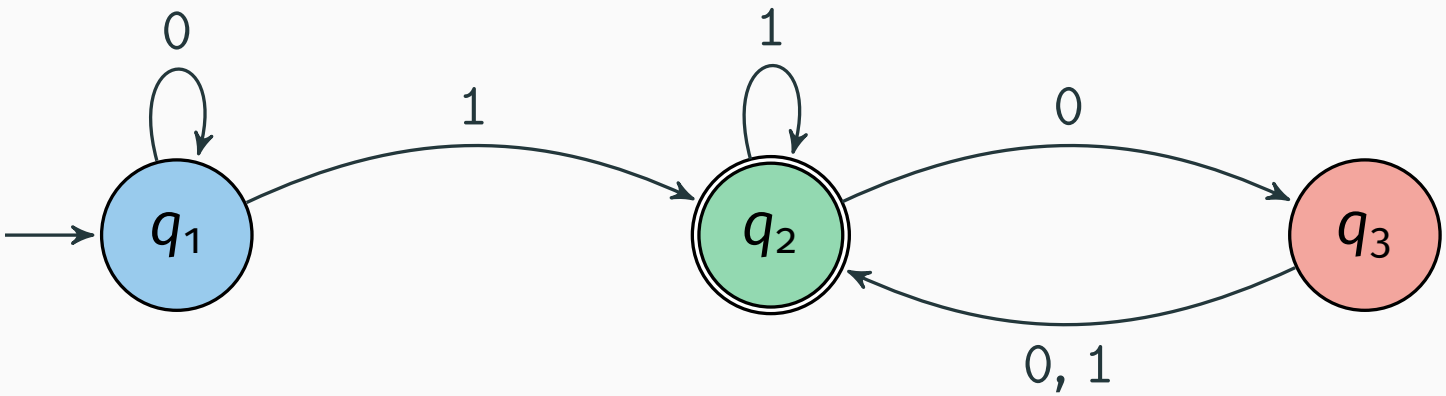

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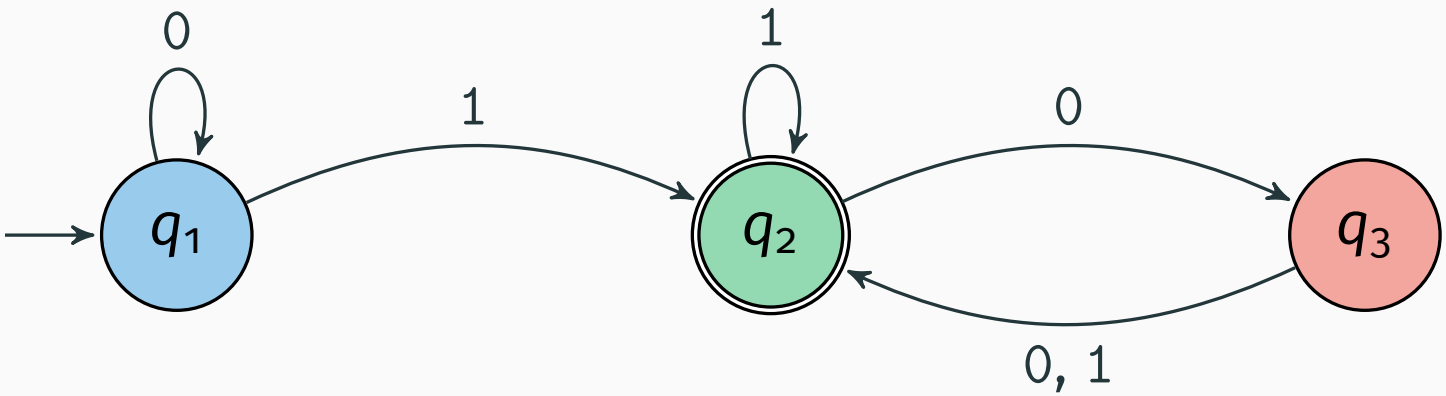

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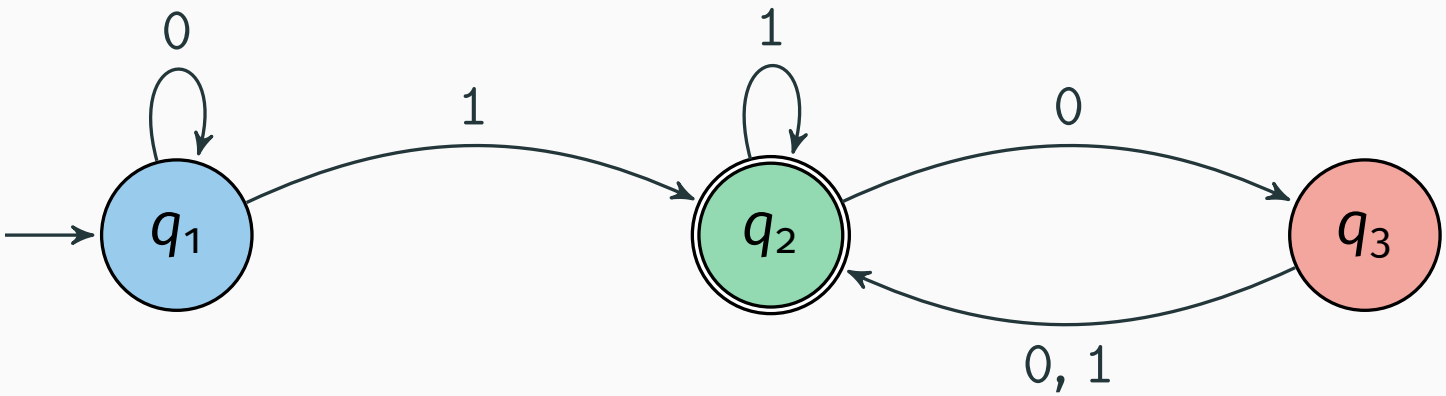

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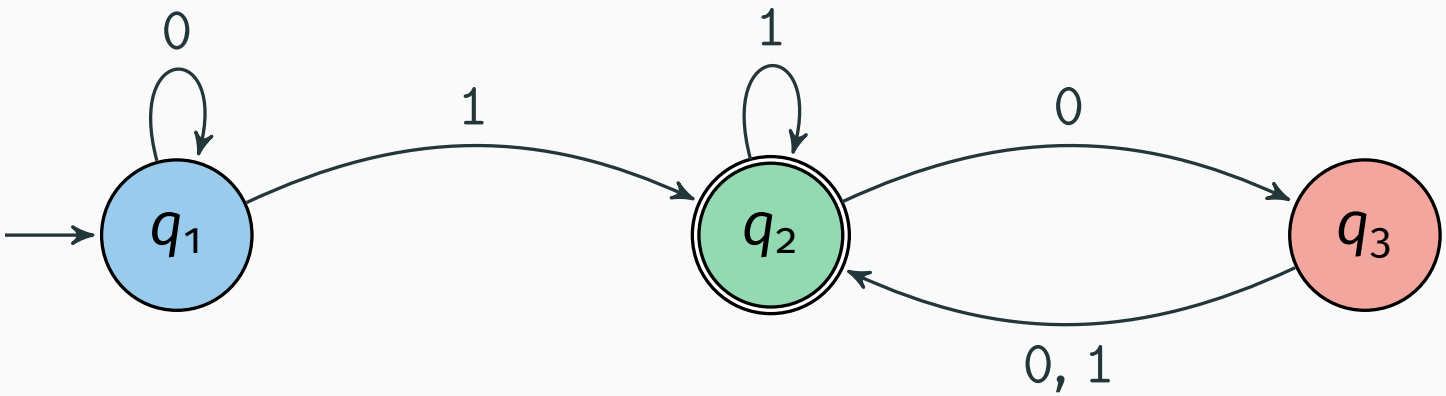

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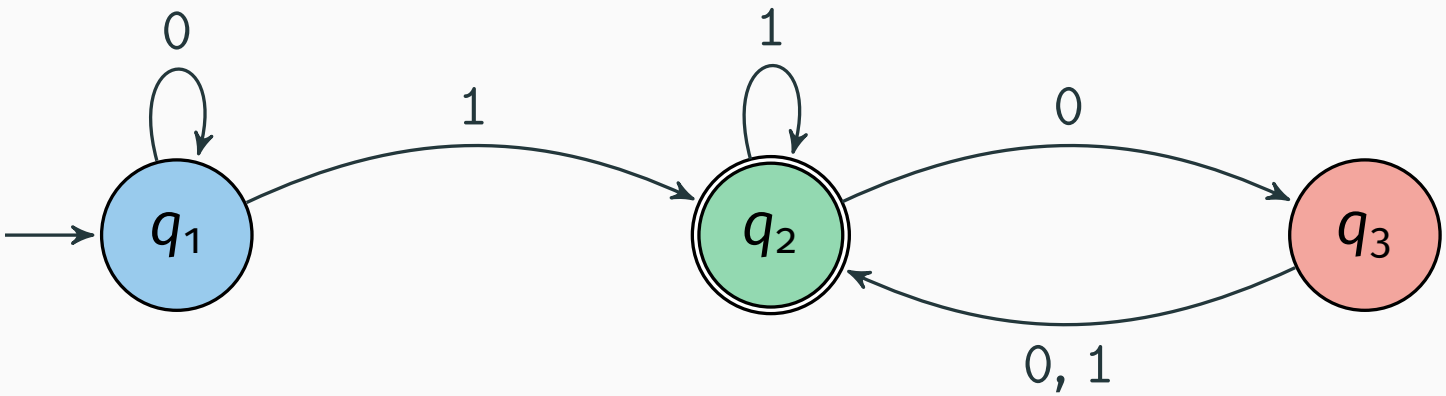

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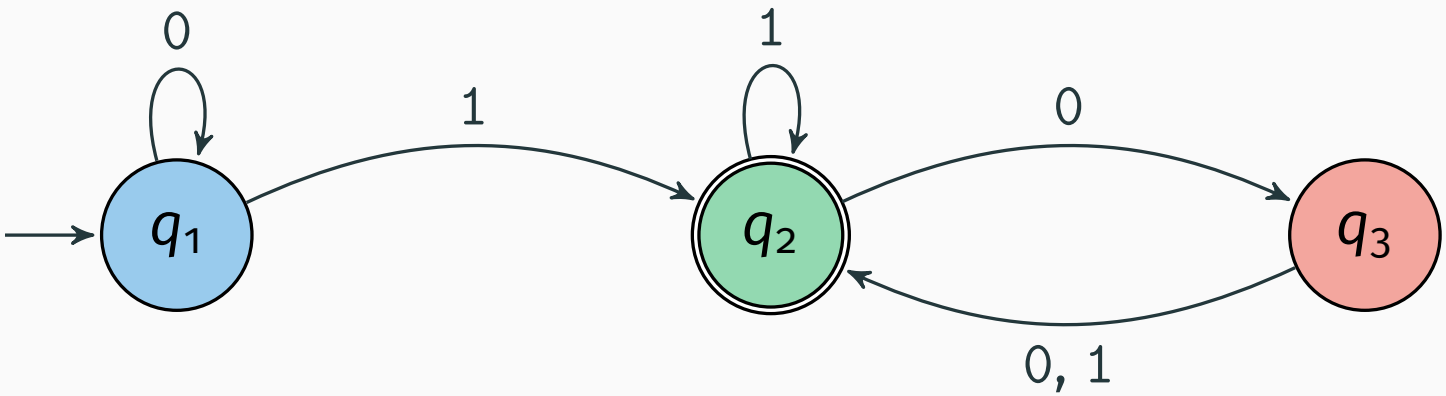

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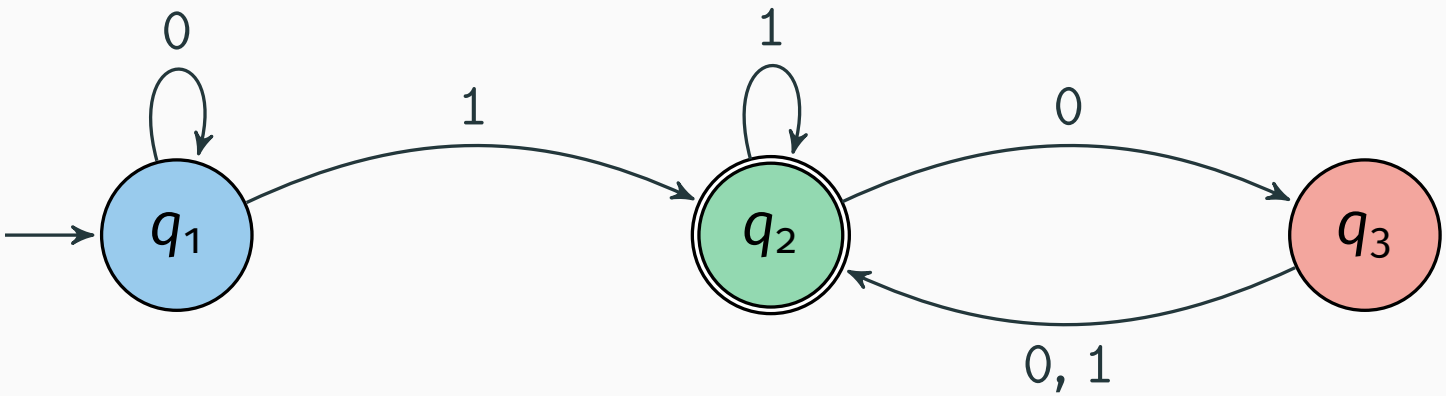

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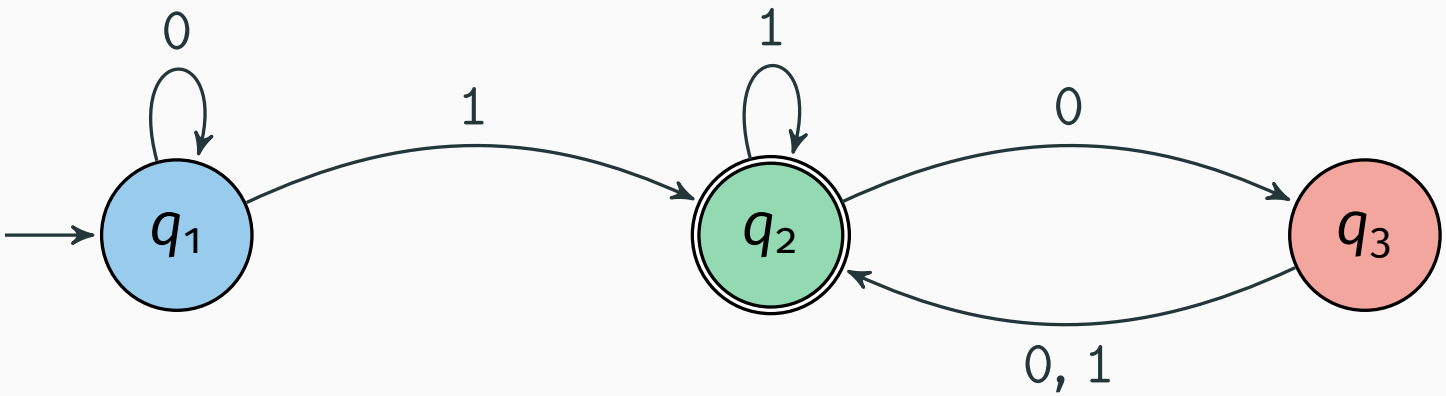

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Question

Which strings are accepted by this automaton?

Words and languages

Definition

- An *alphabet* is a finite set, Σ.
- A Σ-*word* is a finite sequence of elements from Σ.
- For any word w, $|w|$ denotes the *length* of w.
- The *empty sequence* also counts as a word, and is denoted by ϵ.
- Σ^* denotes the (infinite) set of all Σ-words.
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- Idea: a language *is* a computational problem, namely: to determine whether or not a given input word belongs to it.

Words and languages

Examples

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- $\Sigma = \{\texttt{front}, \texttt{rear}, \texttt{both}, \texttt{neither}\}$. Two examples of Σ-words are: (`front`, `front`, `both`, `neither`), and (`neither`, `neither`, `neither`, `neither`, `neither`)
  The second word can be briefly denoted as: $\texttt{neither}^5$.

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- Let $w = a_1 \dots a_n \in \Sigma^*$ be a Σ-word of length n. Then:
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  - a computation is accepting if the last state, q_n, is in F;
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- The language recognized by $\mathcal{A}$ is $L_{\mathcal{A}} := \{w \in \Sigma^* \mid \mathcal{A} \text{ accepts } w\}$.

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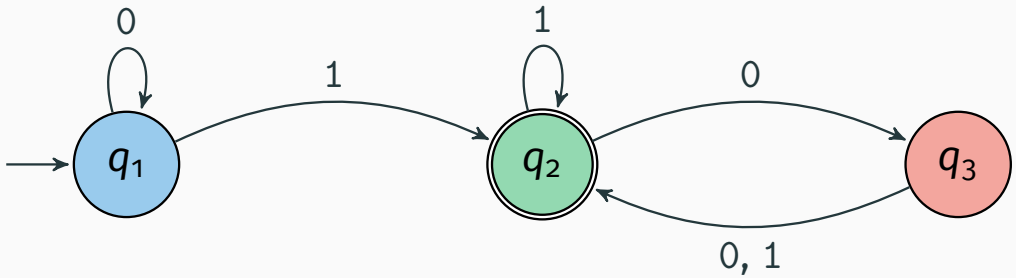

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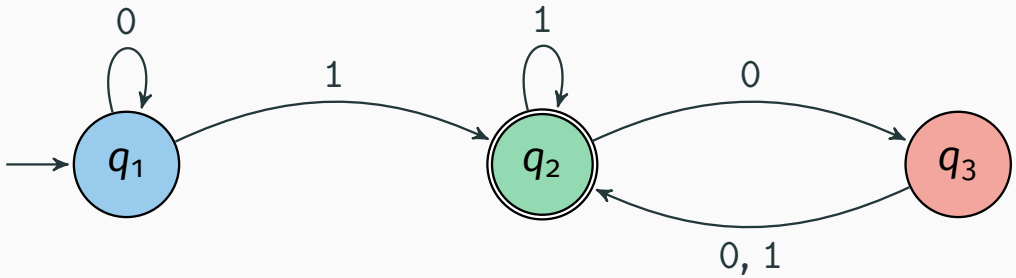

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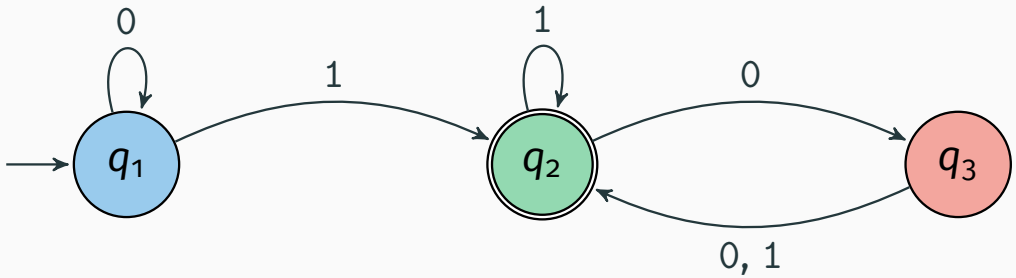

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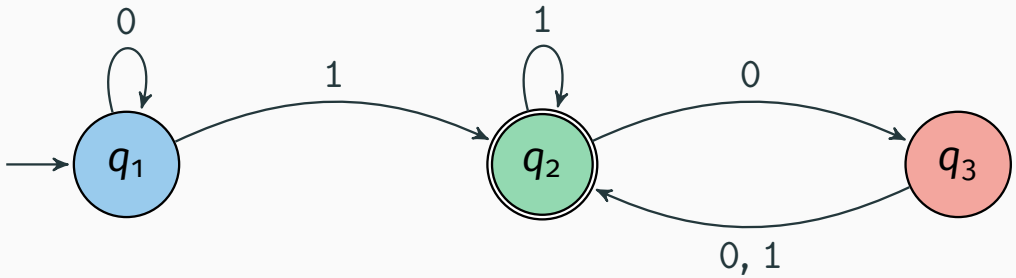

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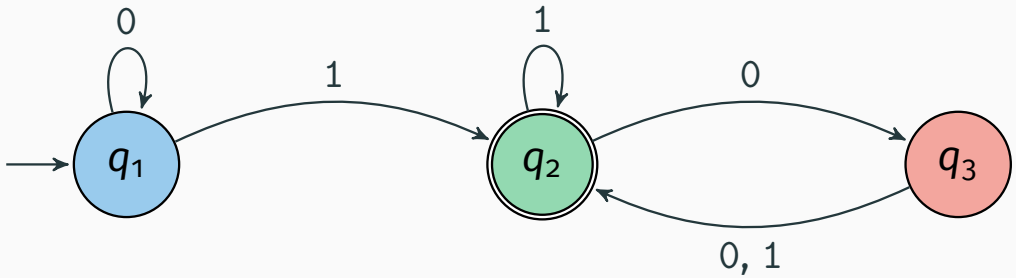

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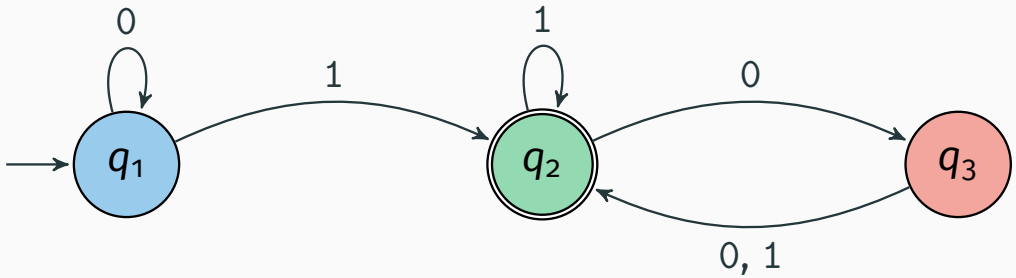

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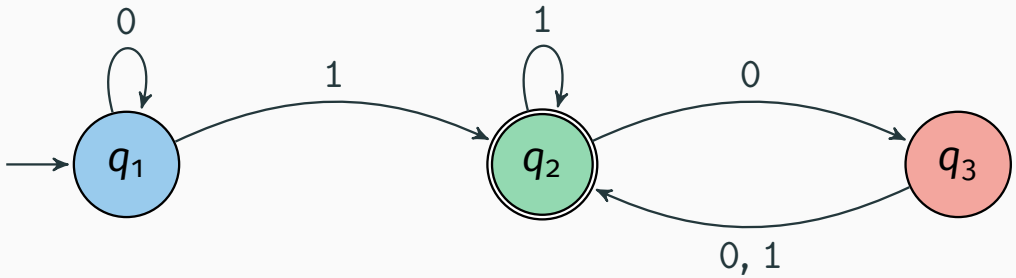

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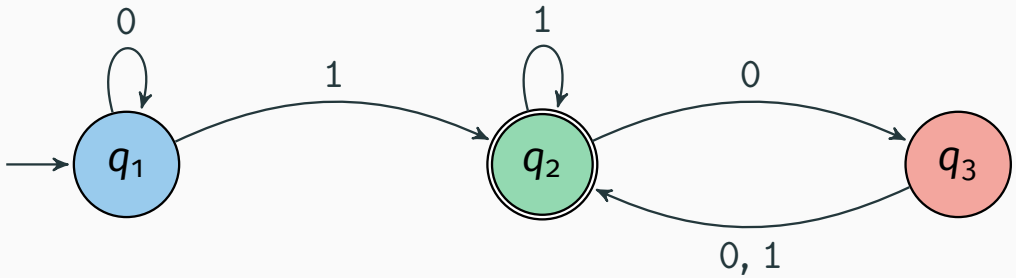

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What is $L_{\mathcal{A}}$?

$L_{\mathcal{A}} = \{w \in \Sigma^* : w$ contains a 1, and an even number of 0's after its last 1 $\}$.

Regular languages and non-determinism

Regular languages

- A language L is called regular if there exists a DFA that recognizes it.

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Examples

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- The language of 'valid passwords': alphanumeric strings containing at least one lower case, one upper case, and one number.
- For any regular expression R, the set of strings which match R.

A nonregular language

Proposition

The following language L is not regular:

$$L := \{w \in \{0,1\}^* \ : \ w \text{ contains as many } 0\text{'s as } 1\text{'s}\}$$

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Proof.

Suppose, towards a contradiction, that $\mathcal{A}$ were a DFA recognizing this language L. Let p be the number of states in $\mathcal{A}$. When $\mathcal{A}$ reads the word 0^p1^p, there must be a state q which is visited twice in the first half of the computation. (Why?)

A nonregular language

Proposition

The following language L is not regular:

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Proof.

Suppose, towards a contradiction, that $\mathcal{A}$ were a DFA recognizing this language L. Let p be the number of states in $\mathcal{A}$. When $\mathcal{A}$ reads the word 0^p1^p, there must be a state q which is visited twice in the first half of the computation. (Why?) Say $\mathcal{A}$ is in state q both after reading 0^k and also after reading 0^{k+r}, where $0 \leq k < p$ and $r > 0$. But then, when $\mathcal{A}$ reads $0^{p+r}1^p$, it will end up in the same last state as when it reads 0^p1^p. This is impossible. (Why?) □

Definition

Let L be a language. A number $p > 0$ is called a pumping length for L if any word $w \in L$ can be factored into three pieces, $w = axb$, with $x \neq \epsilon$ and ax of length $\leq p$, such that $ax^n b \in L$ for all n.

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Lemma (Pumping Lemma)

Any regular language has a pumping length.

This lemma can be used to show that a language is *not* regular: exhibit, for every number $p > 0$, a word w_p in the language that can not be 'pumped'.

Operations on languages

Let Σ be a finite alphabet, and let L_1 and L_2 be Σ-languages.

- The union of L_1 and L_2 is the language

$$L_1 \cup L_2 := \{w \in \Sigma^* \mid w \in L_1 \text{ or } w \in L_2\}.$$

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Question. Suppose L_1 and L_2 are regular. Are $L_1 \cup L_2$, $L_1 \cap L_2$, $\Sigma^* \setminus L_1$, and $L_1 \circ L_2$ also regular?

Union of regular languages

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Proof (sketch).

Let L_1 and L_2 be regular Σ-languages. Choose two DFA's, $\mathcal{A}_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$ and $\mathcal{A}_2 = (Q_2, \Sigma, \delta_2, q_2, F_2)$, recognizing L_1 and L_2, respectively.

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- $Q = Q_1 \times Q_2$;
- $\delta(\langle q_1, q_2 \rangle, a) := \langle \delta_1(q_1, a), \delta_2(q_2, a) \rangle$;
- $q_0 := \langle q_1, q_2 \rangle$;
- $F := (F_1 \times Q_2) \cup (Q_1 \times F_2)$.

Exercise. Check that, indeed, $L_{\mathcal{A}} = L_1 \cup L_2$. □

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We need non-determinism.

Example of a non-deterministic finite automaton

- Consider the language $L := \{w \in \{0,1\}^* :$ the third position from the end of w exists, and is a $1\}$.

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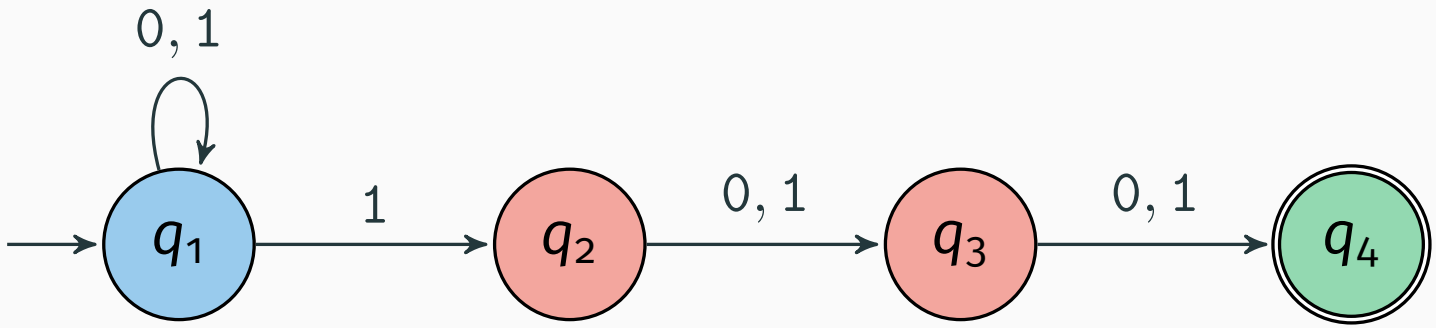

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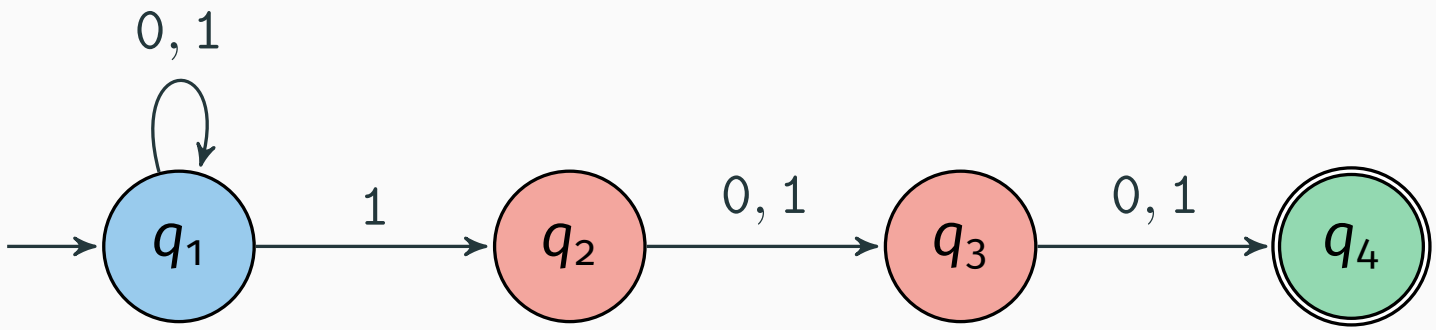

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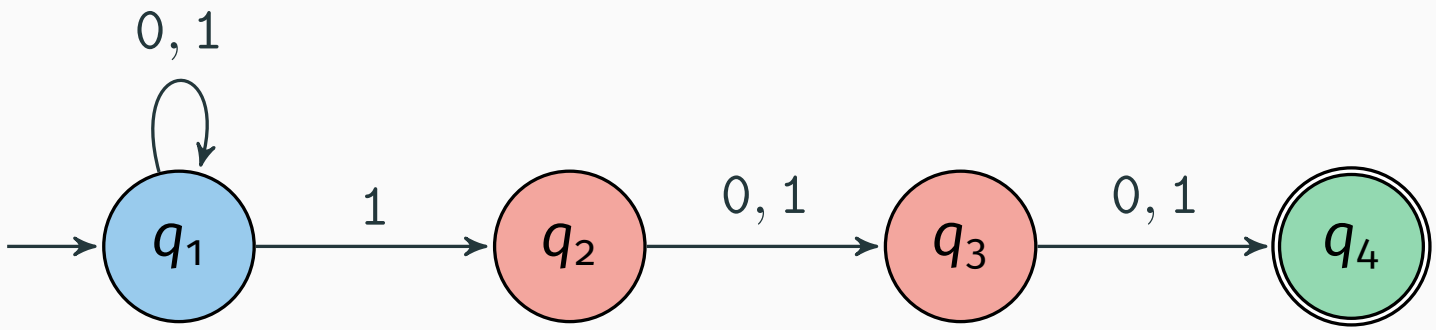

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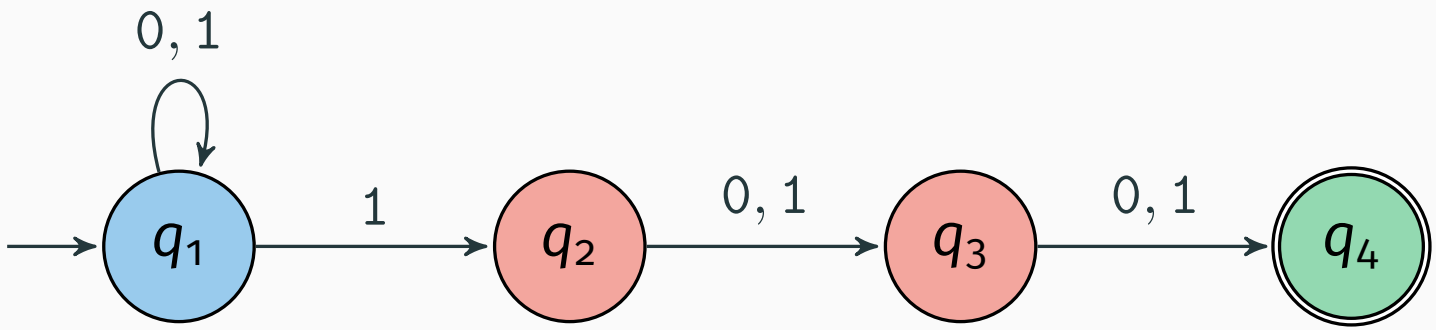

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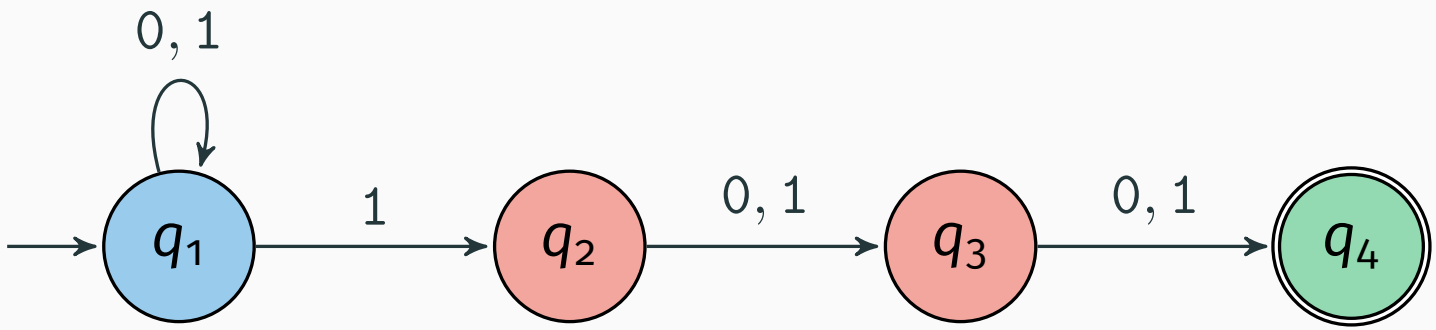

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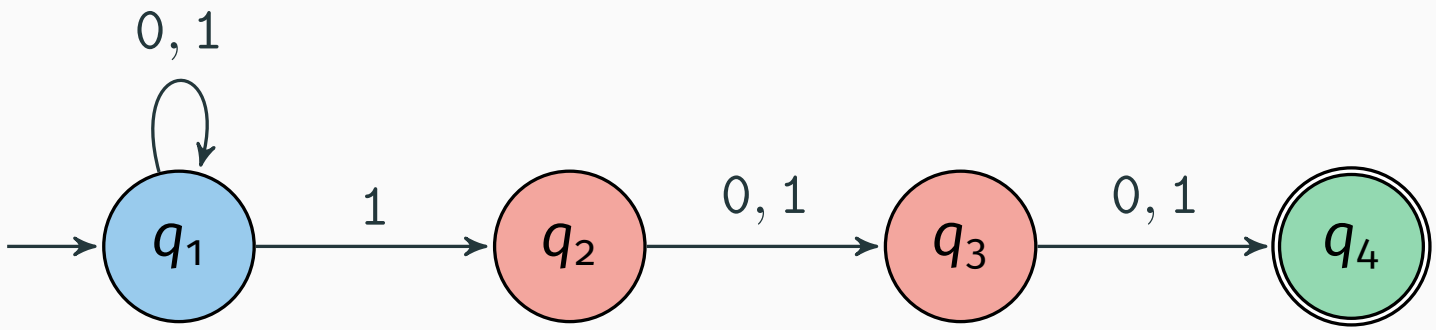

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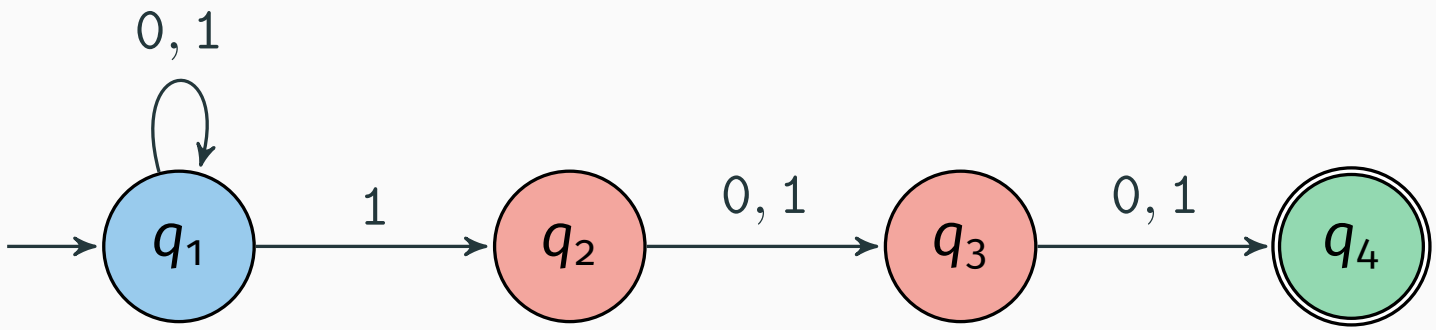

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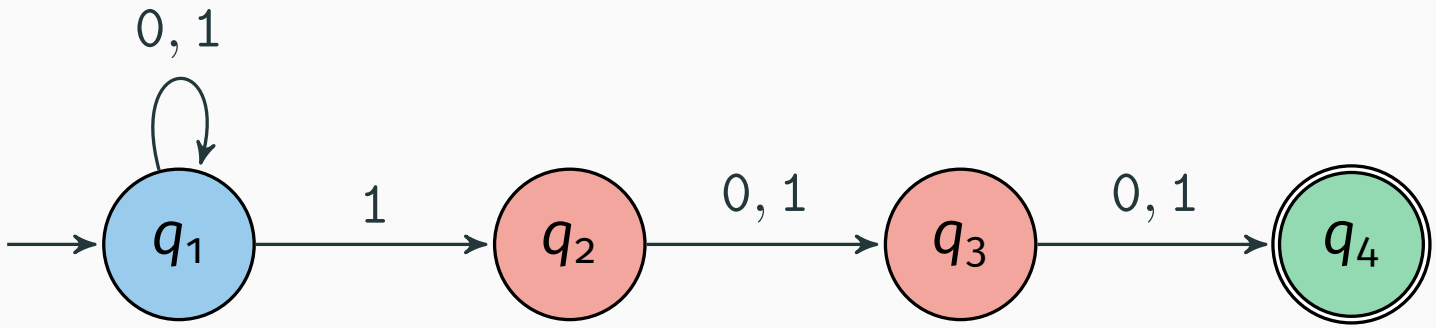

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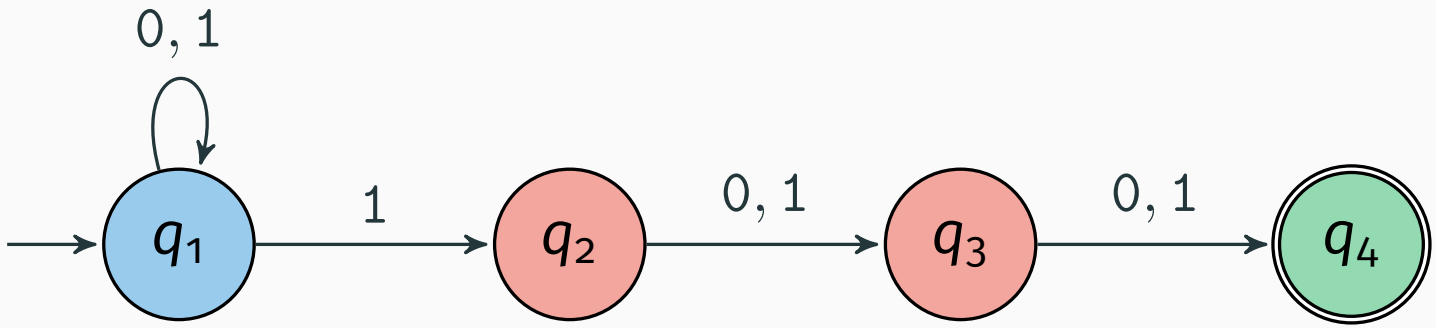

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- A non-deterministic finite automaton (NFA) is a 5-tuple $\mathcal{A} = (Q, \Sigma, \delta, q_0, F)$, where
  - Q is a finite set of states,
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- $q \xrightarrow{a} q'$ means: q is a state in Q, a is a letter in Σ, and $q' \in \delta(q, a)$.
- Let $w = a_1 \dots a_n \in \Sigma^*$ be a Σ-word of length n. Then:
  - a computation of an automaton $\mathcal{A}$ on w is a sequence $q_0 \dots q_n$ of states such that $q_0 \xrightarrow{a_1} q_1 \xrightarrow{a_2} \cdots \xrightarrow{a_n} q_n$;
  - a computation is accepting if the last state, q_n, is in F;
  - $\mathcal{A}$ accepts w if it has some accepting computation on w.
- The language recognized by $\mathcal{A}$ is $L_{\mathcal{A}} := \{w \in \Sigma^* \mid \mathcal{A} \text{ accepts } w\}$.

Concatenation of regular languages

Proposition

The concatenation of two regular languages is regular.

Proof (2nd attempt).

Given two DFA's $\mathcal{A}_1$ and $\mathcal{A}_2$, build one NFA $\mathcal{A}$ by allowing transitions from any final state of $\mathcal{A}_1$ to the initial state of $\mathcal{A}_2$. □

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Proof (sketch).

Given an NFA $\mathcal{A} = (Q, \Sigma, \delta, q_0, F)$, we define a DFA $\mathcal{A}' = (Q', \Sigma, \delta', q_0', F')$ which simulates $\mathcal{A}$, as follows:

- $Q' := \mathcal{P}(Q)$,
- $\delta'(U, a) := \{q' \in Q \mid \text{ there is } q \in U \text{ with } q' \in \delta(q, a)\}$,
- $q_0' := \{q_0\}$,
- $F' := \{U \in Q' \mid \text{ there is } q_f \in U \text{ with } q_f \in F\}$.

Exercise. Check that $\mathcal{A}'$ is deterministic, and $L_{\mathcal{A}'} = L_{\mathcal{A}}$. □

Theorem

A language L can be recognized by an NFA if, and only if, L can be recognized by a DFA.

Monadic Second Order Logic

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- Expressibility in a logic = Computability by a machine.

Logic on words

- Let $w = a_1 \dots a_n$ be a Σ-word.
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Examples of logic descriptions of languages

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> $L = \{w \in \{\texttt{front}, \texttt{rear}, \texttt{neither}, \texttt{both}\}^* \ :$
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Monadic Second Order Logic: syntax

- Basic propositional connectives: $\vee$, $\wedge$, $\neg$, $\ldots$;
- Quantification over first-order variables x, y, $\ldots$ and monadic second-order variables P, Q, $\ldots$;
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The syntax for First Order (FO) Logic is obtained by disallowing second order variables and second order quantifiers.

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- **Exercise.** Shortcuts such as `first`, `last`, `empty`, $\mathtt{S}(x, y)$ ('immediate successor'), ... are definable.

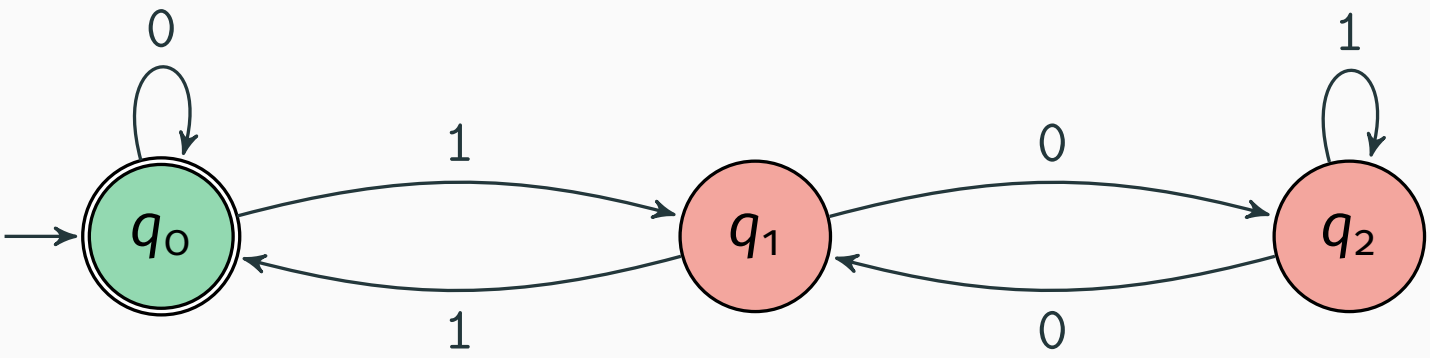
The NFA-MSO Connection

Theorem (Büchi, 1960)

A language is regular if, and only if, it is definable in monadic second order logic:

$$REG = MSO.$$

From an automaton to an MSO-sentence, example



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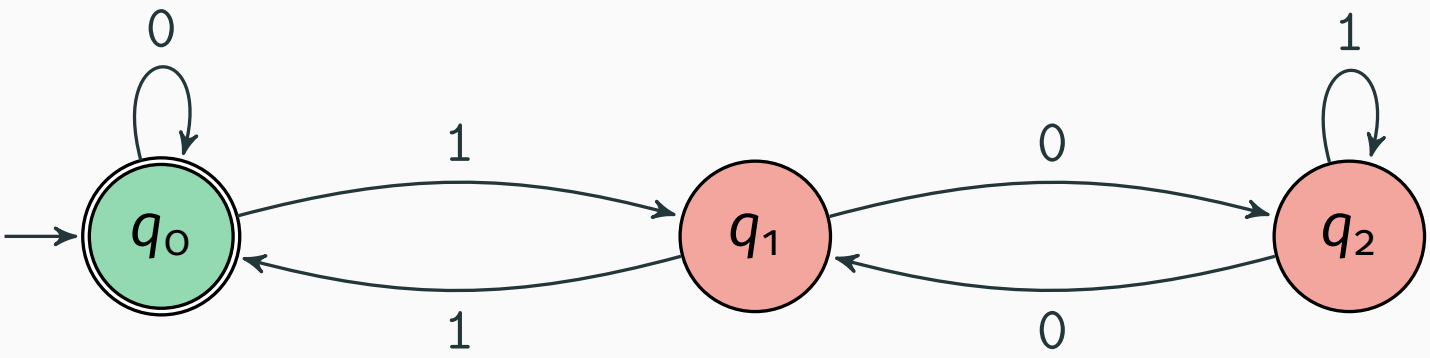

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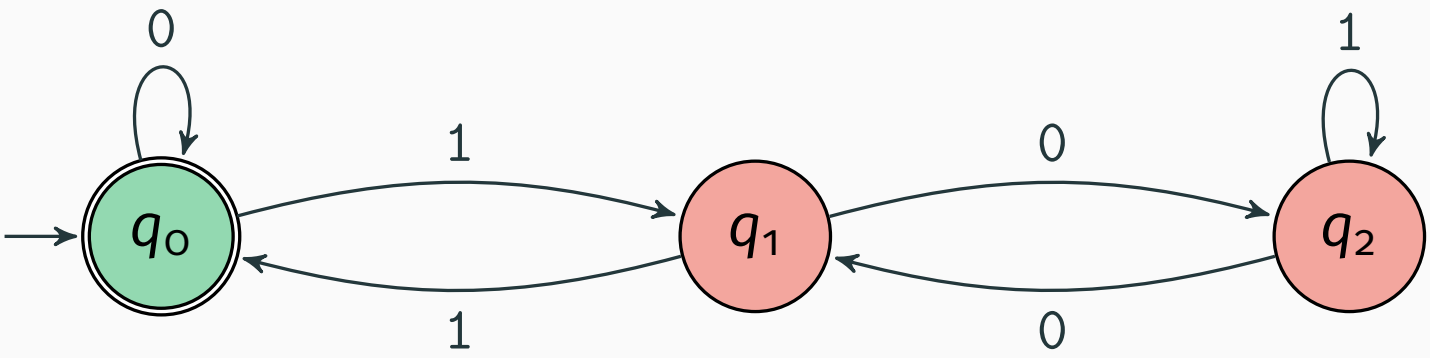

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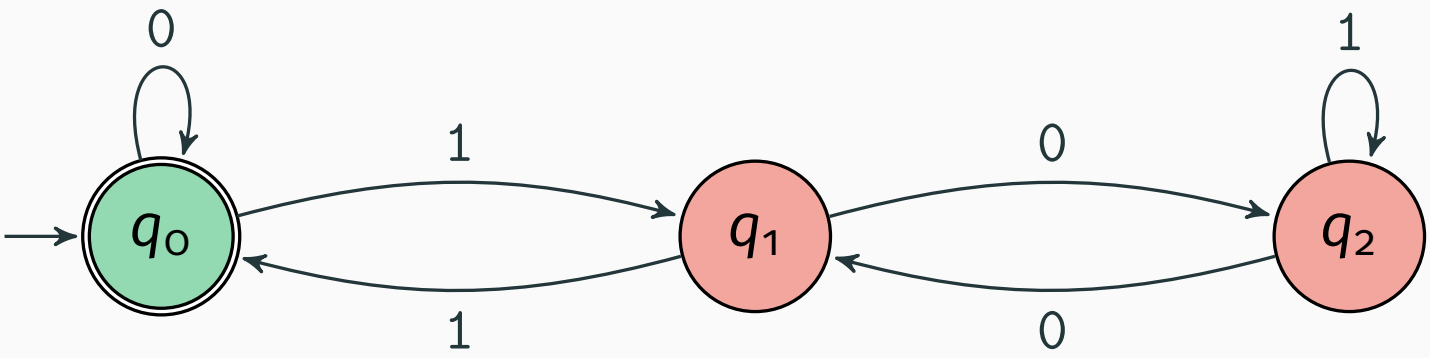

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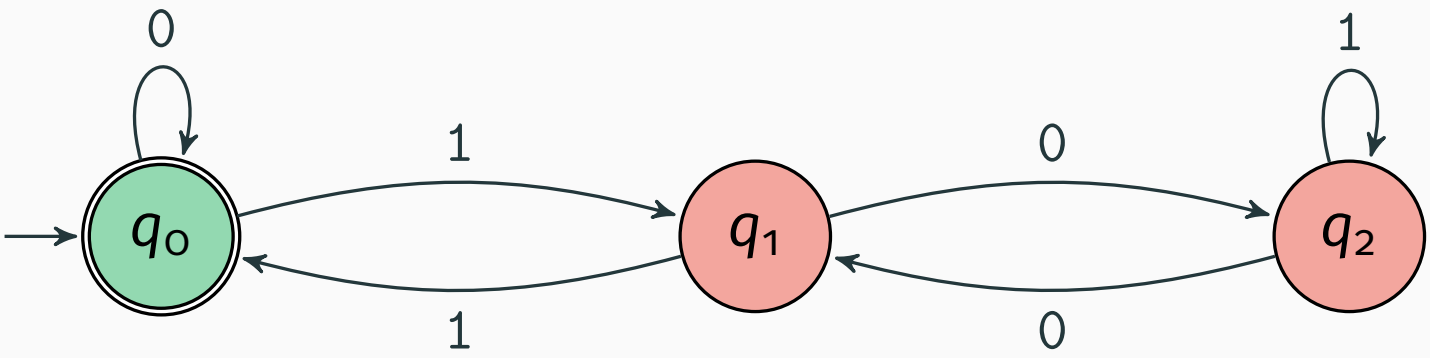

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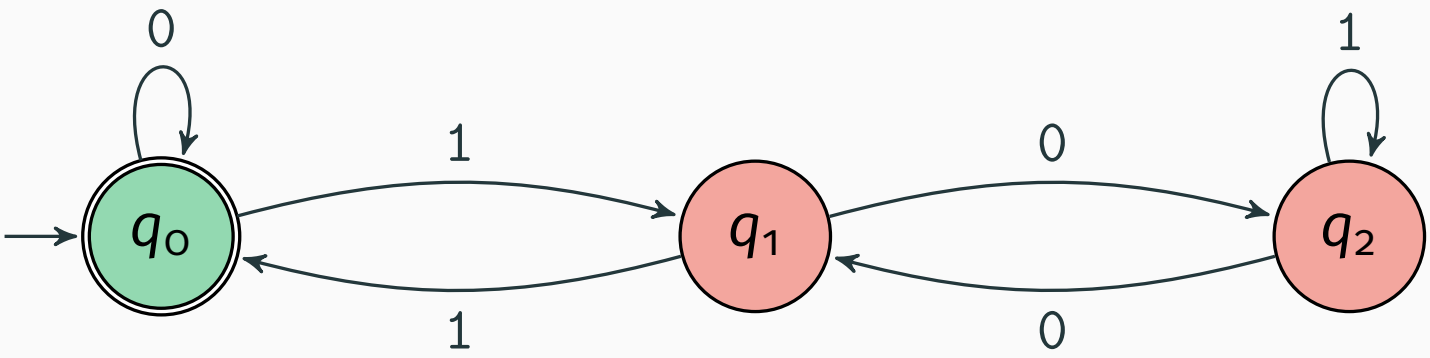

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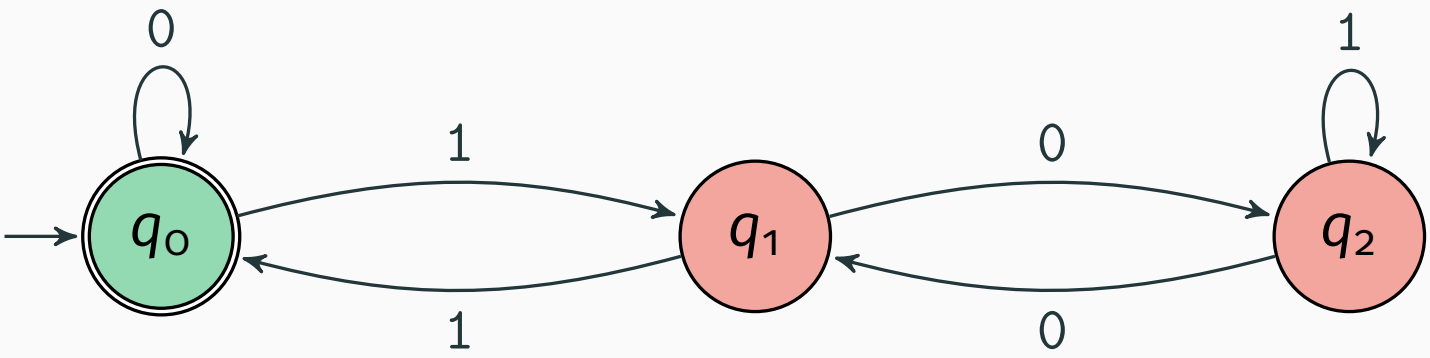

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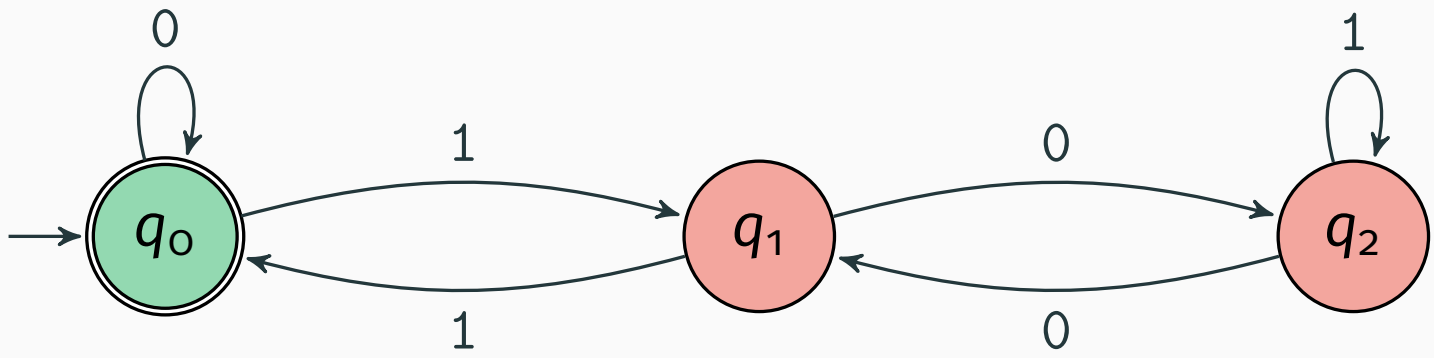

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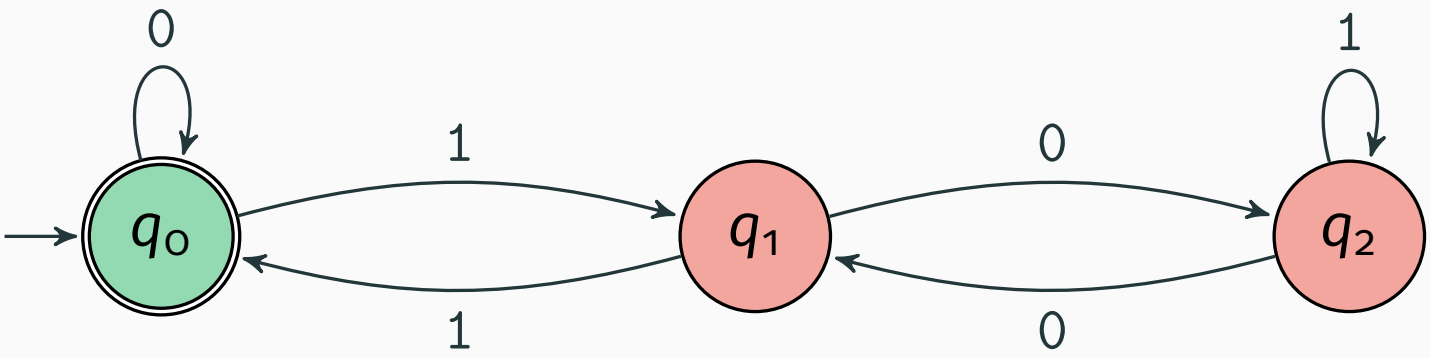

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> **Exercise.** Prove that $\varphi_{\mathcal{A}}$ defines $L_{\mathcal{A}}$.

From an MSO-sentence to an automaton, I

- The proof is by induction on the complexity of the MSO-formula.
- There is a subtlety: an MSO-formula recognizes a language of words *with valuations*.
- If $\mathrm{Var} = \{x_1, \ldots, x_k, P_1, \ldots, P_m\}$ and φ is a formula with free variables in Var, then we need both a word w and a valuation V of Var into the domain of W, before we can determine whether or not φ is true.
- A Σ-word with valuation may be viewed as a $\Sigma \times \mathbf{2}^{\mathrm{Var}}$-word.

Encoding valuations as bit-sequences

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> What happens to first-order variables?

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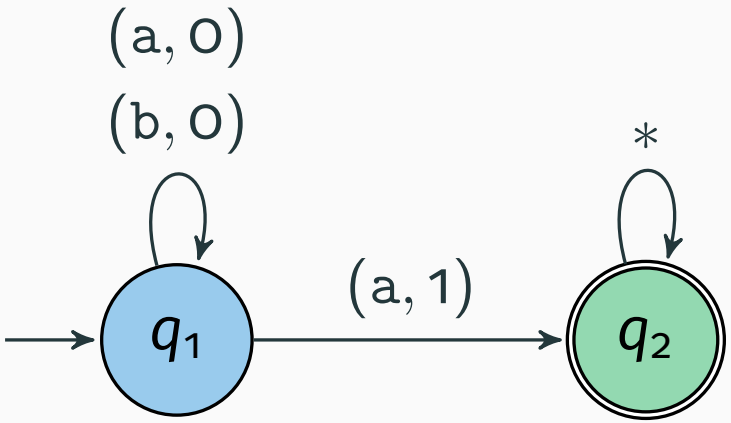

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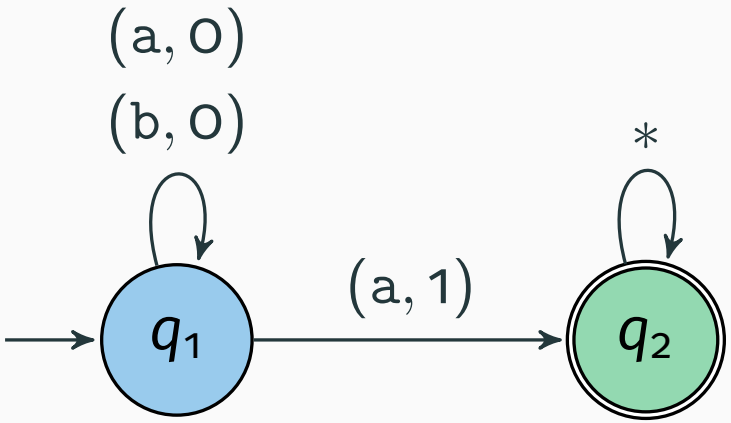

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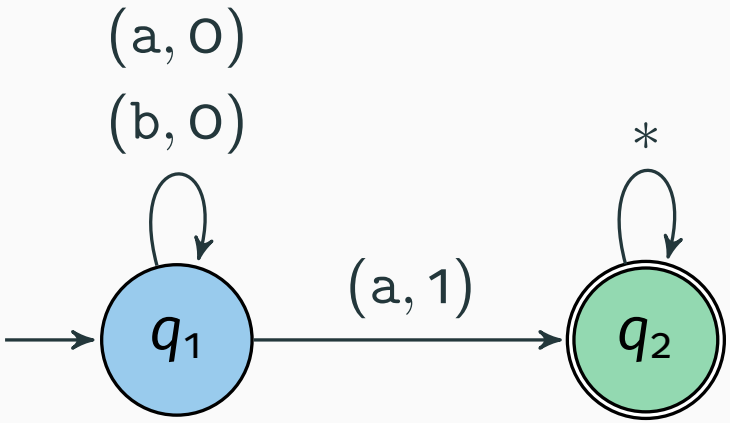

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- Assume given an automaton for φ_1 and φ_2.

> How to get an automaton for $\varphi_1 \vee \varphi_2$? And for $\neg\varphi_1$?

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- Define the NFA $\mathcal{A}' := (Q, \Sigma, \delta', q_0, F)$, where

$$\delta'(q, a) := \delta(q, (a, 0)) \cup \delta(q, (a, 1)).$$

> **Exercise.** Show that $\mathcal{A}'$ accepts an input word w if, and only if, $W \models \exists P \varphi(P)$.

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Summary

- Automata give a first (incomplete) approximation of a model of computation.
- Regular languages are the languages recognized by automata.
- Non-determinism adds flexibility to automata, while not extending the class of regular languages.
- Logic provides an alternative, descriptive formalism for languages.
- Monadic Second Order logic captures exactly the regular languages.

On Wednesday

- Turing machines: motivation and definition.
- Undecidable problems.
- Definition of main complexity classes (P and NP).
- Connection with logic (Fagin's theorem).
- Look at chapters 3 & 4 in Sipser.
- **Reminder!** Top-3 topic preferences due on Blackboard on Wednesday at 5pm.
- **Reminder!** Research question & choice of peer reviewer on Friday at 5pm.

Reference

M. Sipser, *Introduction to the Theory of Computation*, 3rd edition, Cengage Learning (2013).