Turing machines and descriptive complexity

Sam van Gool

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Logic and Computation, Master's course, Utrecht University

Central question in theory of computation

What are the fundamental capabilities

and limitations of computers?

Three types of sub-questions, plus a bonus question

- Complexity theory
  - How *hard* is a computational problem?
  - How much *time*? How much *memory space*?
- Computability theory
  - Is the answer to a problem *computable*, or not?
- Automata theory
  - What is a *minimal* model for computation?
- For each of the above, what is the connection to *logic*?

Overview for today

Turing machines and Decidability

Undecidability

Complexity classes

Describing complexity via logic

Turing machines and Decidability

Turing machines vs. Automata

> What are the essential components of a computer?

A computer (Turing machine) has:

- Initial input
- Ability to read from memory
- Ability to write to memory
- Final (yes/no) output
- Internal states

An automaton has:

- Initial input
- Ability to read from memory:

  once and in one direction
- ~~Ability to write to memory~~
- Final (yes/no) output
- Internal states

Turing machines vs. Automata: four differences

1. A Turing machine can read and write;
2. There is a read/write head that can move right and left;
3. The tape (memory) is infinite;
4. There are two special accept and reject states, which take effect as soon as they are reached.

Turing machine, schematically

Input/Output Tape	1	1	0	0	0	␣	␣	···

Reference: http://www.texample.net/tikz/examples/turing-machine-2/

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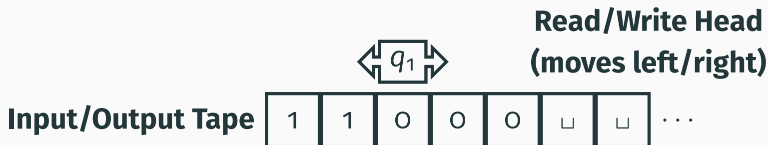

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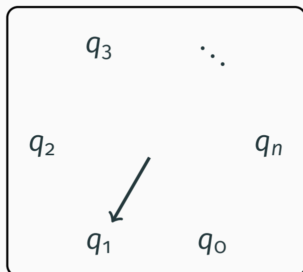
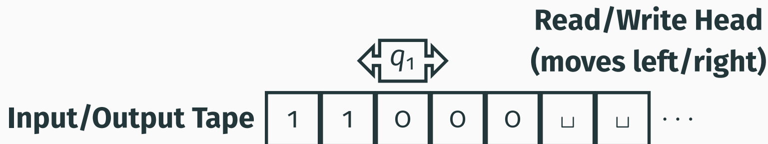


Finite Control



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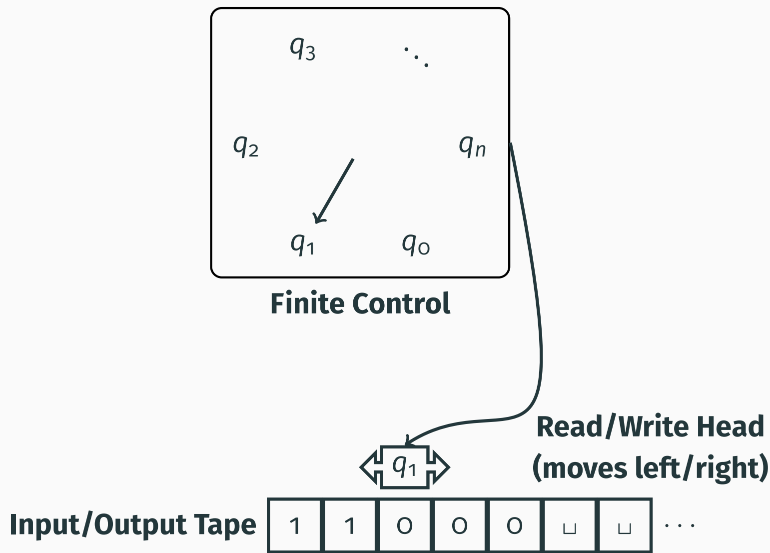

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Turing machine, example

How to compute the language

$$L = \{w\#w \mid w \in \{0,1\}^*\}?$$

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5. Move back left until you encounter '$\#$';
6. Move back left to the last 'x', then move one right and go to step 1.

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7. Move to the right to check that everything is marked until the first ‘$\sqcup$’.
8. Accept (whenever stuck before: reject).

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$\rightarrow$ Simulation

Definition

A *Turing machine* is a 7-tuple, $(Q, \Sigma, \Gamma, \delta, q_0, q_a, q_r)$, where

- Q is a finite set of *states*;
- Σ is a finite *input alphabet* not containing the symbol $\sqcup$;
- Γ is a finite *tape alphabet* containing $\Sigma \cup \{\sqcup\}$;
- $\delta\colon Q \times \Gamma \to Q \times \Gamma \times \{L, R\}$ is a *transition function*;
- $q_0 \in Q$ is the start state;
- $q_a \in Q$ is the accept state;
- $q_r \in Q$ is the reject state; $q_r \neq q_a$.

Accept, reject, halt, loop

Definition

Let $M = (Q, \Sigma, \Gamma, \delta, q_0, q_a, q_r)$ be a Turing machine.

- A *configuration* is a 3-tuple (u, q, v), where $u, v \in \Gamma^*$ and $q \in Q$.

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- We say M *accepts* $w \in \Sigma^*$ if there is a sequence of configurations $C_1, \ldots, C_k$, with $C_1 = (\epsilon, q_0, w)$ and $C_k = (u_k, q_a, v_k)$, where C_i yields C_{i+1} for each $1 \leq i < k$.

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- If M neither accepts nor rejects w, we say it *loops* or *does not halt* on w.

> **Exercise.** Design a Turing machine that does not halt on any input. $\rightarrow$ Solution

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But: the precise notion of Turing machine can make a (big!) difference for efficiency.

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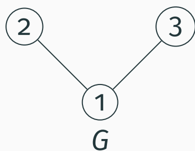

$(1, 2, 3)((1, 2), (1, 3))$

$\langle G \rangle$

$$CONN := \{\langle G \rangle \mid G \text{ is a finite connected graph.}\}$$

> Is the language CONN decidable?

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- One can *implement* this algorithm in a Turing machine.
- One can *prove* that this algorithm always *terminates* and yields the *correct* output on any finite graph.

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- Also decidable: acceptance by an NFA.
- Also decidable if an automaton accepts *any* word at all. (“Emptiness problem”)
- Also decidable if two automata accept *the same* words. (“Equivalence problem”)

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- U only shows that A_{TM} is recursively enumerable.
- The language A_{TM} is not recursive.

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A language L is called recursive or (Turing-)decidable if $L = L(M)$ for some Turing machine M *which halts on all inputs*:

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Undecidability

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It is undecidable whether a Turing machine M halts on an input w.

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- **Problem specification.** ("The Halting Problem")
  - Input: An encoding $\langle (M, w) \rangle$ of a pair (M, w), where M is a Turing machine and w is an input string.
  - Desired output: Accept the encoding $\langle (M, w) \rangle$ if M halts on input w, reject otherwise.

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Theorem (Turing, 1936)

It is undecidable whether a Turing machine M halts on an input w.

- **Problem specification.** (“The Halting Problem”)
  - Input: An encoding $\langle (M, w) \rangle$ of a pair (M, w), where M is a Turing machine and w is an input string.
  - Desired output: Accept the encoding $\langle (M, w) \rangle$ if M halts on input w, reject otherwise.
- Turing’s Theorem: no Turing machine can do this!

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- Suppose, towards a contradiction, that H were a Turing machine which decides the Halting problem: it accepts the encoding of (M, w) if M halts on w, and it rejects the encoding of (M, w) if M loops on w.

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  - If H accepts $\langle(N, n)\rangle$, then N loops on input n;
  - If H rejects $\langle(N, n)\rangle$, then N rejects , so N halts on input n.
- In both cases, H gives the wrong answer on input n. □

No general procedure for bug checks will do.

Now, I won't just assert that, I'll prove it to you.

I will prove that although you might work till you drop,

you cannot tell if computation will stop.

excerpt from: Geoffrey K. Pullum, "Scooping the Loop Snooper" (2000)

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$$\frac{a|b|ca|a|abc}{ab|ca|a|ab|c}.$$

Complexity classes

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- Complexity can be measured in terms of *time* and *space*.
- We only discuss time complexity today.

Definition

Suppose M is a (deterministic or non-deterministic) Turing machine that halts on all inputs.

The *time complexity function* of M is defined as:

$$f_M(n) := \max_{|w| \leq n} \#\text{steps in any computation of } M \text{ on } w.$$

Linear and polynomial time

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- We say M runs in *(deterministic) polynomial time* if there exist numbers k and C such that $f_M(n) \leq Cn^k$ for n sufficiently large.
- We say a language L *can be decided in polynomial time*, or simply "L is in **P**" if there exists a polynomial time deterministic Turing machine which decides L.
- **Cobham's Thesis.** Problems in **P** are 'efficiently solvable' or 'tractable'.

Examples of problems in P

- Given a graph G and two nodes s and t in G, decide if G has a path from s to t.
- Given a graph G and a coloring of the nodes with three colors, decide if the coloring is *correct*, i.e., do adjacent nodes have distinct colors?
- Given an $n \times n$ filled Sudoku grid, decide if it is filled out correctly.
- Any linear programming problem.
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  (Shown to be in **P** in 2002.)

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- We say a language L *can be decided in non-deterministic polynomial time*, or simply "L is in **NP**" if there exists a polynomial time non-deterministic Turing machine which decides L.
- Since deterministic Turing machines are a special case of non-deterministic Turing machines, obviously $\mathbf{P} \subseteq \mathbf{NP}$.

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- **SAT.** Given a propositional formula (in conjunctive normal form), is it *satisfiable*?
- In fact, these problems are all **NP**-*complete* ("hardest").
- The fact that **SAT** is NP-complete is known as the *Cook-Levin Theorem* (1970's).

P vs. NP

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Describing complexity via logic

Recall Büchi's Theorem:

A language is regular if, and only if, it is definable in monadic second order logic:

$$\mathsf{REG} = \mathsf{MSO}.$$

Describing a language in existential second-order logic

- Consider the language

$$L := \{\langle G\rangle \mid G = (V, E) \text{ admits a correct 3-coloring}\}.$$

- This language can be *described* by an existential second-order sentence:

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$$\exists R \exists Y \exists B \big(\forall x [Rx \vee Yx \vee Bx \wedge \forall y [Exy \rightarrow \\ (\neg(Rx \wedge Ry) \wedge \neg(Bx \wedge By) \wedge \neg(Yx \wedge Yy))].$$

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Two main differences when moving from *monadic* to *existential* second-order logic:

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Formalisms like ESO are used in database query languages like SQL.

Theorem (Fagin, 1973)

Existential second-order logic exactly captures the complexity class **NP***:*

$$ESO = \mathbf{NP}.$$

Proof that ESO $\subseteq$ NP

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- Thus, M decides the ESO-sentence Φ in polynomial time.

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- There is an ESO sentence of the form

$$\exists C_1 \cdots \exists C_n \psi$$

  which expresses that M reaches an accepting state within n^k steps.
- (Many details left to check.)

A logic for P?

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- **Immerman-Vardi Theorem.** First order logic with induction (in the form of *least fixed points*) exactly captures **P** *on ordered structures*.
- The problem of finding a logic that captures **P** is open.
- $\rightarrow$ M. Grohe, “The quest for a logic capturing PTIME”, LICS (2008).

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- Undecidability of the Halting problem.
- Main time complexity classes: P and NP.
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Homework

- **Reminder!** Submit your top-3 paper topic preferences **today** before 5pm.
- **Reminder!** On Friday before 5pm, submit:
  1. Your research question;
  2. Name of a preferred peer reviewer;
  3. Preference for presenting early/middle/late.
- **Self-study**: review this week's material, ask questions if needed.

Reference

M. Sipser, *Introduction to the Theory of Computation*, 3rd edition, Cengage Learning (2013).