On Priestley duality and sheaf representations for MV-algebras

Sam van Gool
(joint work with Mai Gehrke and Vincenzo Marra)

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Infinite-valued logics and algebraic semantics

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- MV-algebras (C.C. Chang, 1958): equivalent algebraic semantics for Ł.
- The Basic Logic BL and BL-algebras (P. Hajek, 1996): common generalization of several fuzzy propositional logics, among which Ł, and equivalent algebraic semantics.

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- This definition is equivalent to Chang's original axiomatization, but emphasizes the underlying lattice structure.

Typical MV-algebras

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- Free MV-algebras;

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- Boolean algebras.

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- Our main result.

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  - The embedding $\mathrm{Free}_n \hookrightarrow C([0,1]^n, [0,1])$ depends on the choice of generators.
- If A is a Boolean algebra, then Z_A is its Stone dual space.
- If A is any chain, then Z_A is a one-point space.
- If A has infinitesimals, then we do not have $A \hookrightarrow C(Z_A, [0,1])$.

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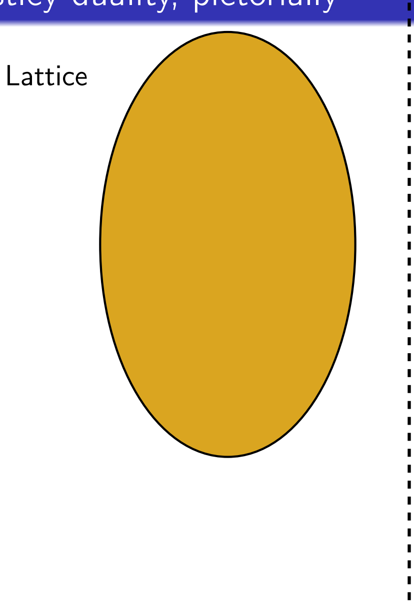
- Its dual Priestley space X comes with a ternary relation, or coalgebra structure $X \times X \to \mathcal{P}(X)$.
- For MV-algebras, this relation can be reduced to a binary operation $X \times X \to X$ on the dual space.

Priestley duality, pictorially



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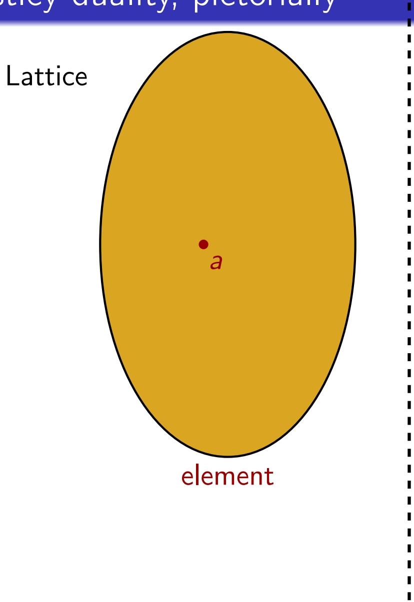

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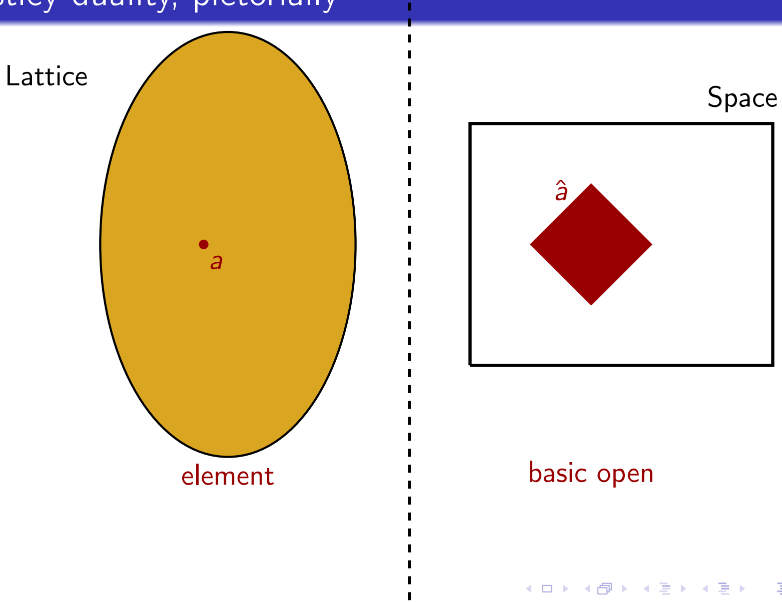

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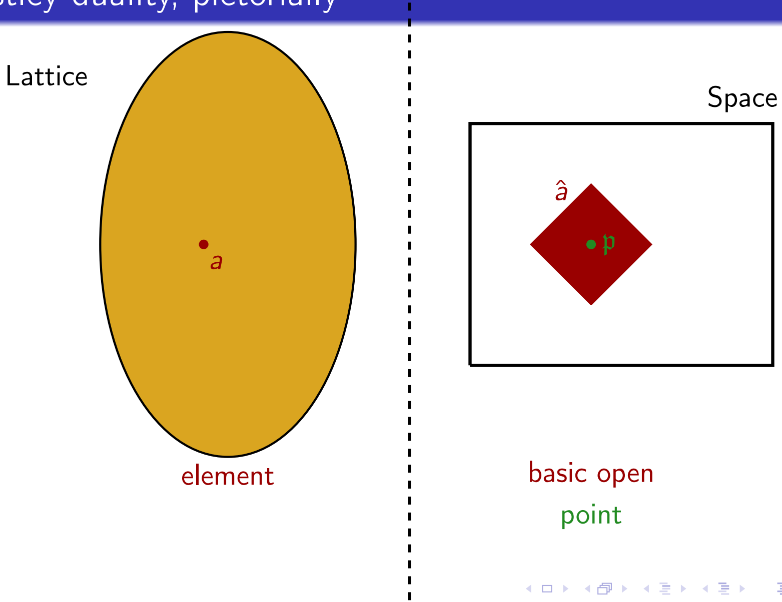

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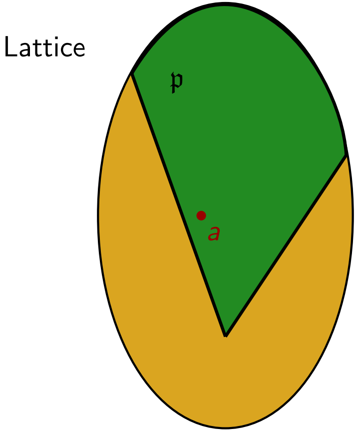
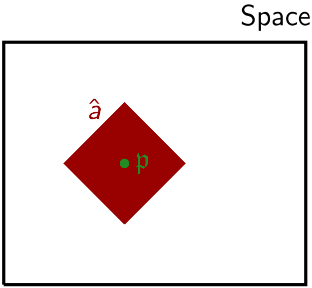
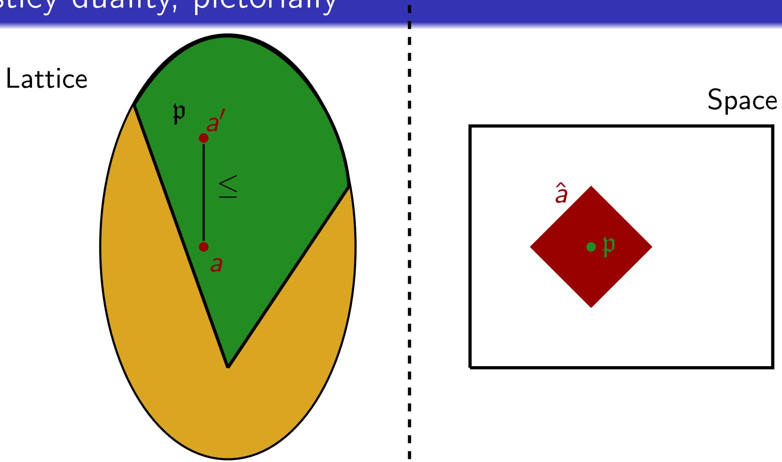




element
prime filter

basic open
point

Priestley duality, pictorially



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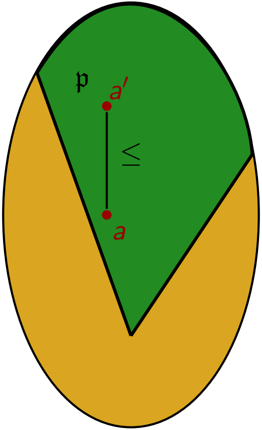
Priestley duality, pictorially

Lattice



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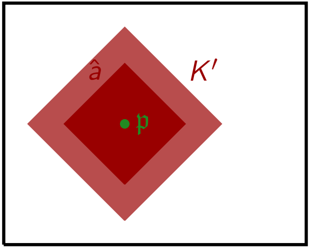
Space



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subset inclusion

MV-algebras as sheaves of chains

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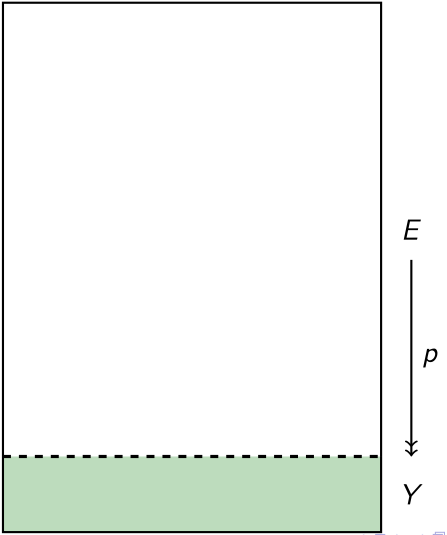
- Result (Dubuc & Poveda, 2010): the original MV-algebra is isomorphic to the algebra of global sections of this sheaf.
- Other sheaf representations: Filipoiu & Georgescu (1995), Ferraioli & Lettieri (2011).

Sheaf representation, pictorially



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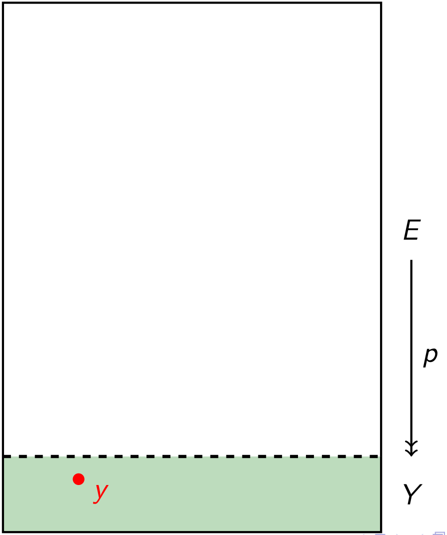

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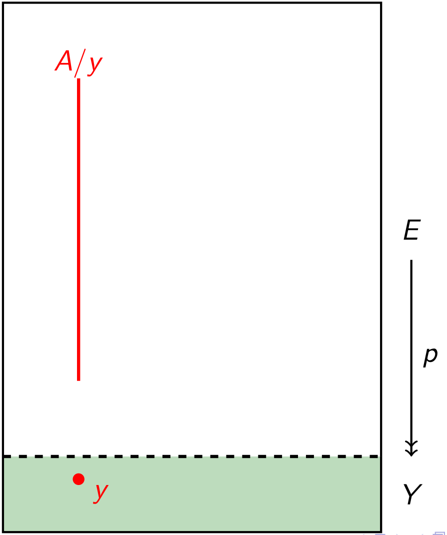

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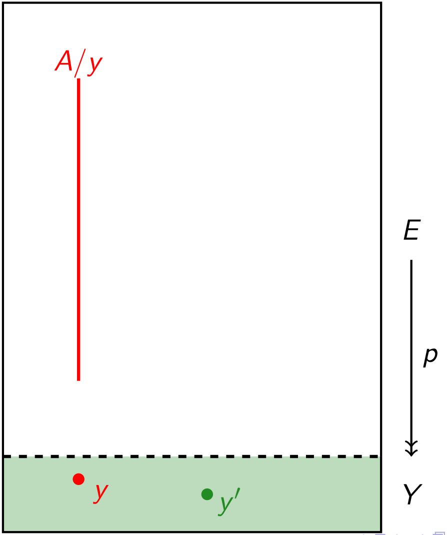

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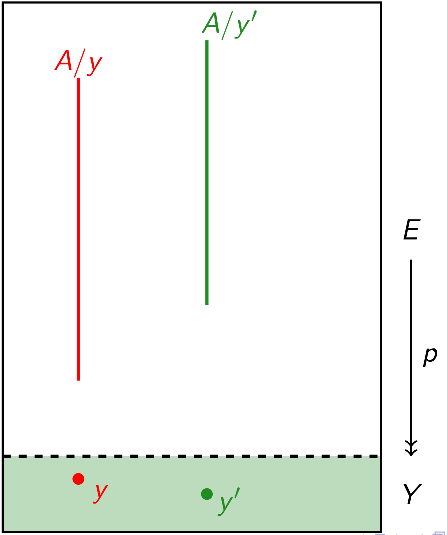

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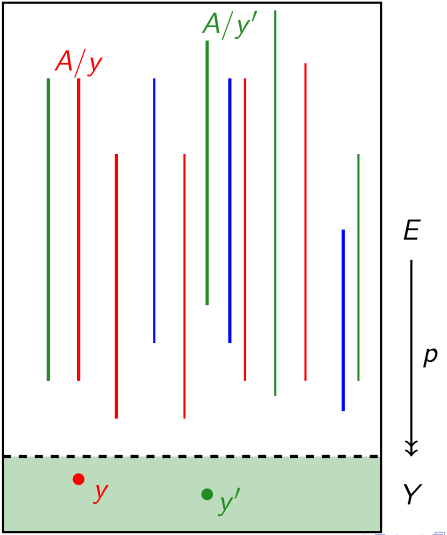

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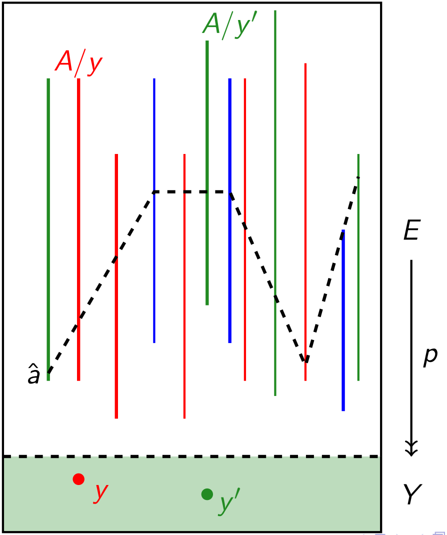

Our main result

Theorem

The Priestley dual space of an MV-algebra A decomposes as a spectral sum over the space of prime MV ideals of A.

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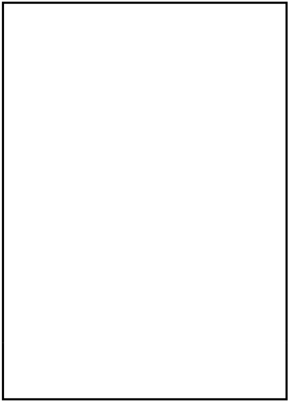

$X = A_*$

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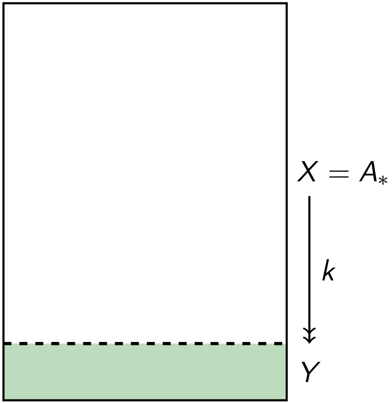

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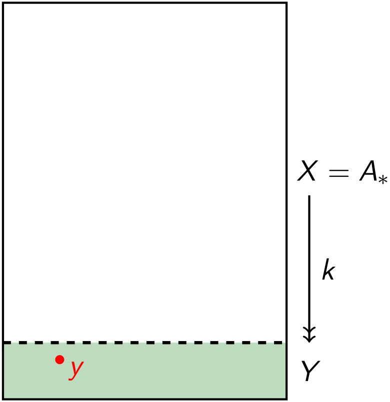

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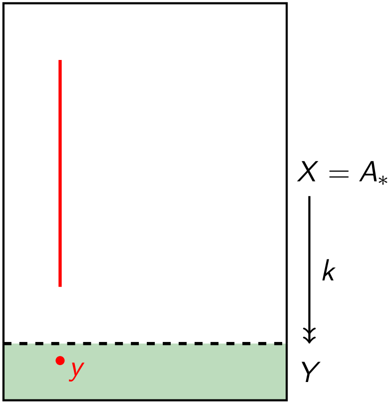

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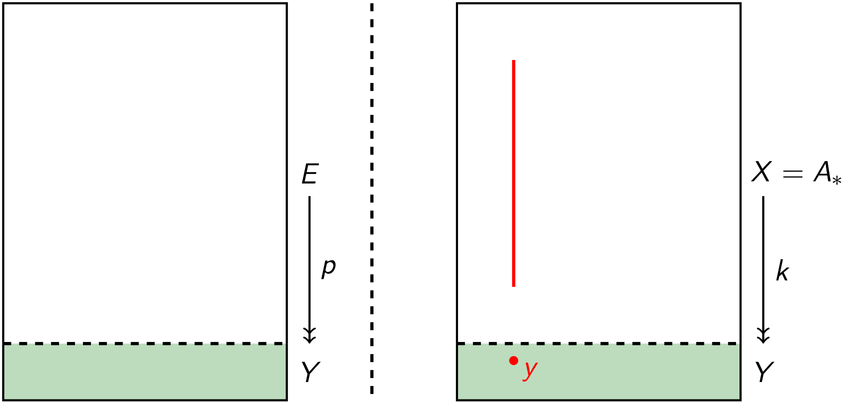

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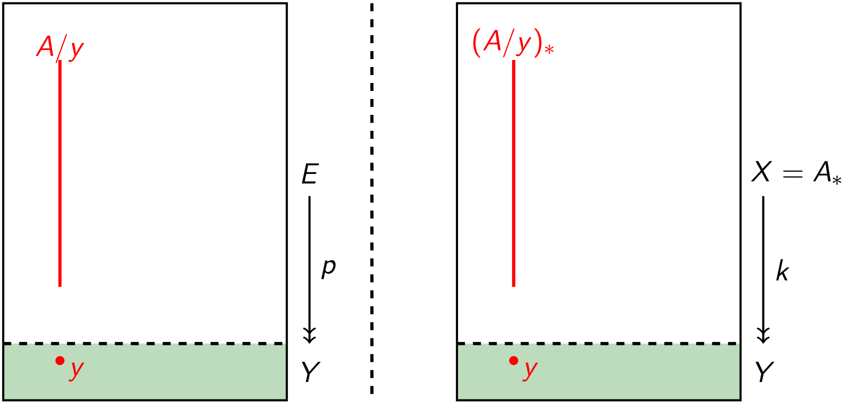

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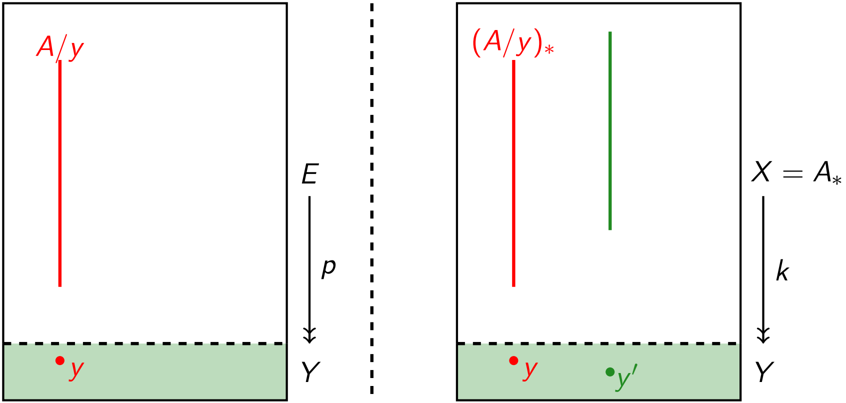

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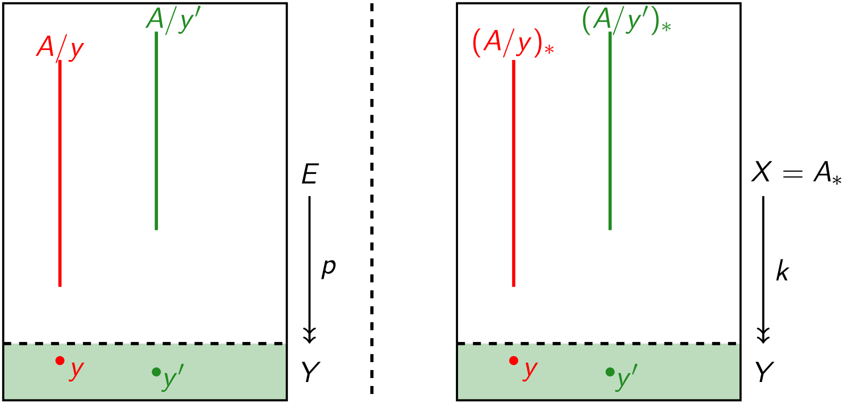

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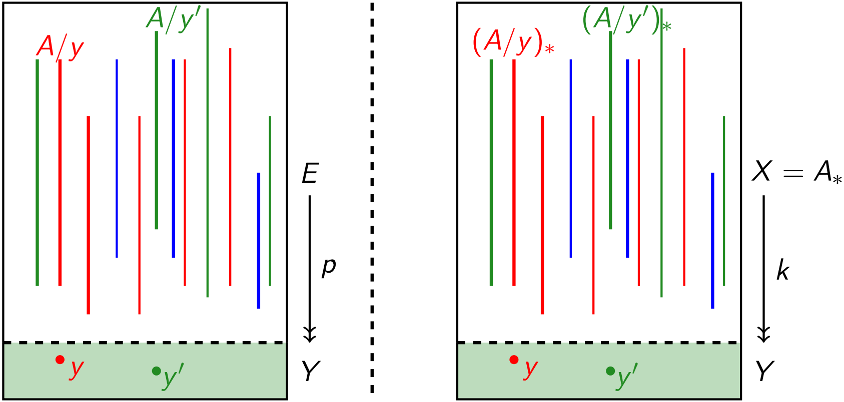

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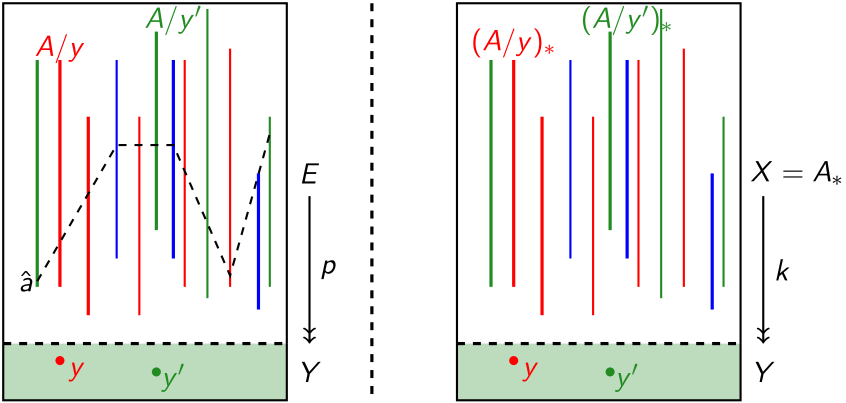

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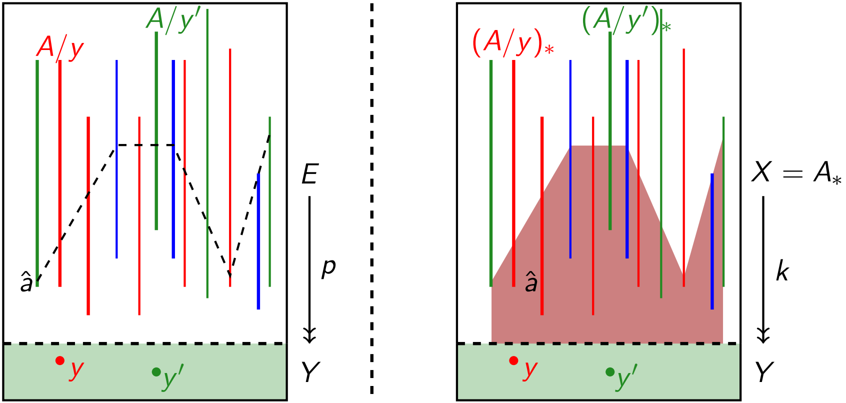

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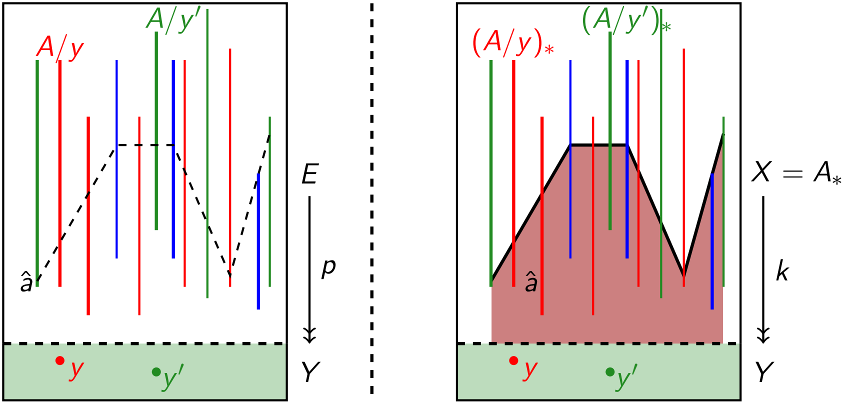

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- New proofs for sheaf representations in one common framework

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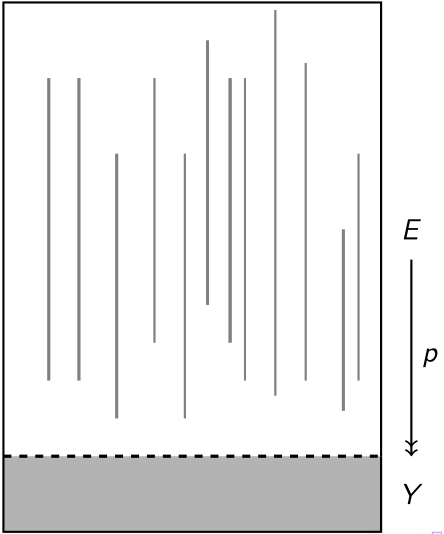
- To find: an element $b \in A$ such that $b|_{U_i} = a_i|_{U_i}$.

Illustration



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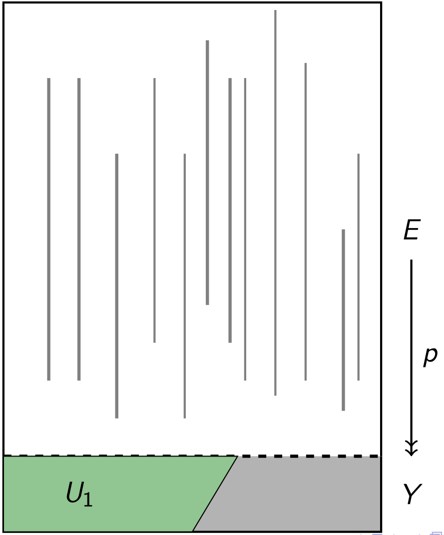

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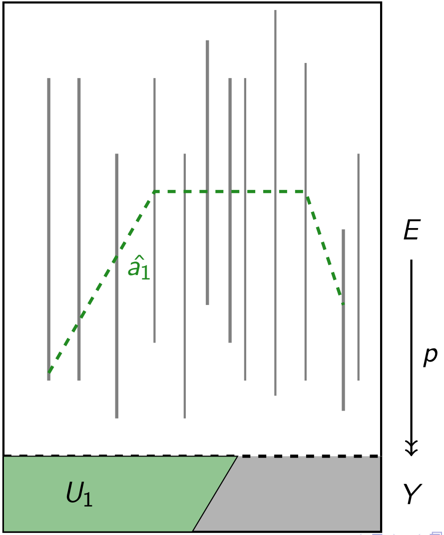

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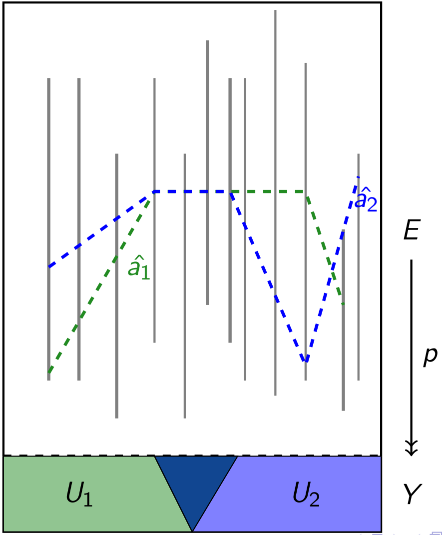

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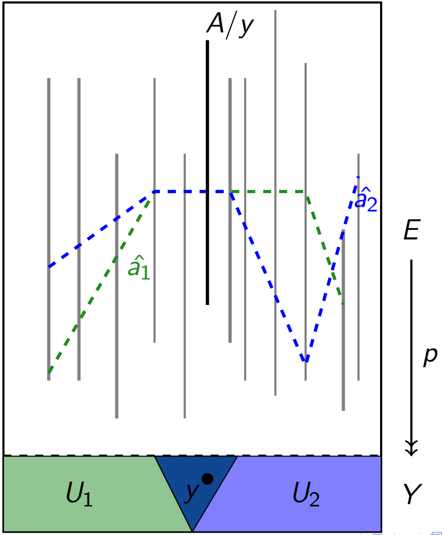

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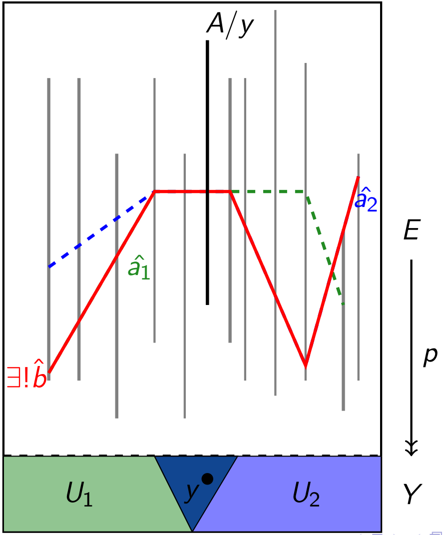

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- To prove: K is a clopen downset of the dual space.
- This also yields a formula for b:

$$b = \bigvee_{i=1}^{n} (a_i \odot \neg m u_i),$$

where $m, n \in \mathbb{N}$ and $u_i \in A$ such that $\hat{u}_i = U_i$.

Classical statement

Theorem (Kaplansky 1959)

Let S_1, S_2 be compact Hausdorff spaces such that the lattices $C(S_1, [0, 1])$ and $C(S_2, [0, 1])$ are isomorphic. Then S_1 and S_2 are homeomorphic.

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If A_1 and A_2 are MV-algebras having isomorphic lattice reducts, then the maximal MV-spectra of A_1 and A_2 are homeomorphic.

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Proof (sketch).

The maximal MV-spectrum can be reconstructed from the lattice spectrum. □

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- Duality provides spatial methods to obtain known and new results for general MV-algebras.

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- Dually: when do coalgebras admit sum decompositions?

On Priestley duality and sheaf representations for MV-algebras

Sam van Gool
(joint work with Mai Gehrke and Vincenzo Marra)

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