Free algebras for Gödel-Löb provability logic

Sam van Gool

University of Bern

AiML
8 August 2014
Groningen

Gödel-Löb provability logic

- GL: extension of basic modal logic K by the Löb axiom

$$\Box(\Box p \to p) \to \Box p \tag{L}$$

Gödel-Löb provability logic

- GL: extension of basic modal logic K by the Löb axiom

$$\Box(\Box p \to p) \to \Box p \tag{L}$$

- Intended interpretation of $\Box$: 'provable in T', where T is a sufficiently strong theory of arithmetic, e.g., PA.

Gödel-Löb provability logic

- GL: extension of basic modal logic K by the Löb axiom

$$\Box(\Box p \to p) \to \Box p \tag{L}$$

- Intended interpretation of $\Box$: 'provable in T', where T is a sufficiently strong theory of arithmetic, e.g., PA.
- The logic GL is sound and complete with respect to finite irreflexive transitive models.

Gödel-Löb provability logic

- GL: extension of basic modal logic K by the Löb axiom

$$\Box(\Box p \to p) \to \Box p \tag{L}$$

- Intended interpretation of $\Box$: ‘provable in T’, where T is a sufficiently strong theory of arithmetic, e.g., PA.
- The logic GL is sound and complete with respect to finite irreflexive transitive models.
- The logic GL is not canonical: (L) does not hold in the frame of maximal consistent sets.

Free algebras

- A GL-algebra (also: Magari algebra) is a pair $(B, \Box)$, where B is a Boolean algebra and $\Box$ is a unary operation such that $\Box\top \approx \top$, $\Box(a \wedge b) \approx \Box a \wedge \Box b$, and $\Box(\Box a \to a) \to \Box a \approx \top$.

Free algebras

- A GL-algebra (also: Magari algebra) is a pair $(B, \Box)$, where B is a Boolean algebra and $\Box$ is a unary operation such that $\Box\top \approx \top$, $\Box(a \wedge b) \approx \Box a \wedge \Box b$, and $\Box(\Box a \to a) \to \Box a \approx \top$.
- The free n-generated GL-algebra, $F(p_1, \dots, p_n)$, is the algebra of GL-equivalence classes of modal formulas in propositional variables $p_1, \dots, p_n$.

Free algebras

- A GL-algebra (also: Magari algebra) is a pair $(B, \Box)$, where B is a Boolean algebra and $\Box$ is a unary operation such that $\Box\top \approx \top$, $\Box(a \wedge b) \approx \Box a \wedge \Box b$, and $\Box(\Box a \to a) \to \Box a \approx \top$.
- The free n-generated GL-algebra, $F(p_1, \ldots, p_n)$, is the algebra of GL-equivalence classes of modal formulas in propositional variables $p_1, \ldots, p_n$.
- Aim. Give a construction of the free n-generated GL-algebra.

Step construction of free algebras

- This is an instance of a general approach (cf. Ghilardi 1995; Coumans & vG 2013; Ghilardi, Bezhanishvili & Jibladze 2013).

Step construction of free algebras

- This is an instance of a general approach (cf. Ghilardi 1995; Coumans & vG 2013; Ghilardi, Bezhanishvili & Jibladze 2013).
- Idea: given the equivalence classes of modal formulas up to $\Box$-rank n, and the action of $\Box$ on them, uniformly describe how to obtain the equivalence classes of modal formulas up to $\Box$-rank $n+1$.

Step construction of free algebras

- This is an instance of a general approach (cf. Ghilardi 1995; Coumans & vG 2013; Ghilardi, Bezhanishvili & Jibladze 2013).
- Idea: given the equivalence classes of modal formulas up to $\Box$-rank n, and the action of $\Box$ on them, uniformly describe how to obtain the equivalence classes of modal formulas up to $\Box$-rank $n + 1$.
- If all axioms of the logic have $\Box$-rank 1, one can always obtain such a uniform description in a straightforward way.

Step construction of free algebras

- This is an instance of a general approach (cf. Ghilardi 1995; Coumans & vG 2013; Ghilardi, Bezhanishvili & Jibladze 2013).
- Idea: given the equivalence classes of modal formulas up to $\Box$-rank n, and the action of $\Box$ on them, uniformly describe how to obtain the equivalence classes of modal formulas up to $\Box$-rank $n+1$.
- If all axioms of the logic have $\Box$-rank 1, one can always obtain such a uniform description in a straightforward way.
- For more complicated axioms, a new ingredient is needed: partial or one-step algebras.

Step construction of free algebras

- This is an instance of a general approach (cf. Ghilardi 1995; Coumans & vG 2013; Ghilardi, Bezhanishvili & Jibladze 2013).
- Idea: given the equivalence classes of modal formulas up to $\Box$-rank n, and the action of $\Box$ on them, uniformly describe how to obtain the equivalence classes of modal formulas up to $\Box$-rank $n+1$.
- If all axioms of the logic have $\Box$-rank 1, one can always obtain such a uniform description in a straightforward way.
- For more complicated axioms, a new ingredient is needed: partial or one-step algebras.
- Important (open) problem: when does this method work?

Step construction of free algebras

- This is an instance of a general approach (cf. Ghilardi 1995; Coumans & vG 2013; Ghilardi, Bezhanishvili & Jibladze 2013).
- Idea: given the equivalence classes of modal formulas up to $\Box$-rank n, and the action of $\Box$ on them, uniformly describe how to obtain the equivalence classes of modal formulas up to $\Box$-rank $n+1$.
- If all axioms of the logic have $\Box$-rank 1, one can always obtain such a uniform description in a straightforward way.
- For more complicated axioms, a new ingredient is needed: partial or one-step algebras.
- Important (open) problem: when does this method work?
  - Not always: it implies finite model property;

Step construction of free algebras

- This is an instance of a general approach (cf. Ghilardi 1995; Coumans & vG 2013; Ghilardi, Bezhanishvili & Jibladze 2013).
- Idea: given the equivalence classes of modal formulas up to $\Box$-rank n, and the action of $\Box$ on them, uniformly describe how to obtain the equivalence classes of modal formulas up to $\Box$-rank $n+1$.
- If all axioms of the logic have $\Box$-rank 1, one can always obtain such a uniform description in a straightforward way.
- For more complicated axioms, a new ingredient is needed: partial or one-step algebras.
- Important (open) problem: when does this method work?
    - Not always: it implies finite model property;
    - At least for stable logics (cf. Nick Bezhanishvili's talk).

Partial GL-algebras

- A partial modal algebra is a tuple $(A, B, \Box)$, where A is a Boolean algebra, B is a Boolean subalgebra of A, and $\Box : B \to A$ is a function preserving $\wedge$, 1.

Partial GL-algebras

- A partial modal algebra is a tuple $(A, B, \Box)$, where A is a Boolean algebra, B is a Boolean subalgebra of A, and $\Box : B \to A$ is a function preserving $\wedge$, 1.
- (Note: Partial algebras $\cong$ one-step algebras with injective i)

Partial GL-algebras

- A partial modal algebra is a tuple $(A, B, \Box)$, where A is a Boolean algebra, B is a Boolean subalgebra of A, and $\Box : B \to A$ is a function preserving $\wedge$, 1.
- (Note: Partial algebras $\cong$ one-step algebras with injective i)
- $(A, B, \Box)$ is a partial GL-algebra provided that, for any $a, b \in B$:

$$\text{if } b \leq \Box a \to a \text{ then } \Box b \leq \Box a.$$

Partial GL-algebras

- A partial modal algebra is a tuple $(A, B, \Box)$, where A is a Boolean algebra, B is a Boolean subalgebra of A, and $\Box : B \to A$ is a function preserving $\wedge$, 1.
- (Note: Partial algebras $\cong$ one-step algebras with injective i)
- $(A, B, \Box)$ is a partial GL-algebra provided that, for any $a, b \in B$:

$$\text{if } b \leq \Box a \to a \text{ then } \Box b \leq \Box a.$$

- Example: the free rank 0-1 partial GL-algebra $(A_1, A_0, \Box_0)$:

Partial GL-algebras

- A partial modal algebra is a tuple $(A, B, \Box)$, where A is a Boolean algebra, B is a Boolean subalgebra of A, and $\Box : B \to A$ is a function preserving $\wedge$, 1.
- (Note: Partial algebras $\cong$ one-step algebras with injective i)
- $(A, B, \Box)$ is a partial GL-algebra provided that, for any $a, b \in B$:

$$\text{if } b \leq \Box a \to a \text{ then } \Box b \leq \Box a.$$

- Example: the free rank 0-1 partial GL-algebra $(A_1, A_0, \Box_0)$:
  - A_1 is the algebra of GL-equivalence classes of modal formulas of rank ≤ 1,

Partial GL-algebras

- A partial modal algebra is a tuple $(A, B, \Box)$, where A is a Boolean algebra, B is a Boolean subalgebra of A, and $\Box : B \to A$ is a function preserving $\wedge$, 1.
- (Note: Partial algebras $\cong$ one-step algebras with injective i)
- $(A, B, \Box)$ is a partial GL-algebra provided that, for any $a, b \in B$:

$$\text{if } b \leq \Box a \to a \text{ then } \Box b \leq \Box a.$$

- Example: the free rank 0-1 partial GL-algebra $(A_1, A_0, \Box_0)$:
  - A_1 is the algebra of GL-equivalence classes of modal formulas of rank ≤ 1,
  - A_0 is the subalgebra of formulas of rank 0,

Partial GL-algebras

- A partial modal algebra is a tuple $(A, B, \Box)$, where A is a Boolean algebra, B is a Boolean subalgebra of A, and $\Box : B \to A$ is a function preserving $\wedge$, 1.
- (Note: Partial algebras $\cong$ one-step algebras with injective i)
- $(A, B, \Box)$ is a partial GL-algebra provided that, for any $a, b \in B$:

$$\text{if } b \leq \Box a \to a \text{ then } \Box b \leq \Box a.$$

- Example: the free rank 0-1 partial GL-algebra $(A_1, A_0, \Box_0)$:
  - A_1 is the algebra of GL-equivalence classes of modal formulas of rank ≤ 1,
  - A_0 is the subalgebra of formulas of rank 0,
  - $\Box_0 : A_0 \to A_1$ is defined in the obvious way.

Free image-total extension

- Let $(A, B, \Box)$ be a partial modal algebra. We say that a homomorphism $i : (A, B, \Box) \to F(A, B, \Box)$ into a partial GL-algebra $F(A, B, \Box)$ is a free image-total GL-extension if for any homomorphism $h : (A, B, \Box) \to (C, D, \Box)$ into a partial GL-algebra such that $h(B) \subseteq D$, there exists a unique $\bar{h} : F(A, B, \Box) \to (C, D, \Box)$ such that $\bar{h} \circ i = h$.

Free image-total extension

- Let $(A, B, \Box)$ be a partial modal algebra. We say that a homomorphism $i : (A, B, \Box) \to F(A, B, \Box)$ into a partial GL-algebra $F(A, B, \Box)$ is a free image-total GL-extension if for any homomorphism $h : (A, B, \Box) \to (C, D, \Box)$ into a partial GL-algebra such that $h(B) \subseteq D$, there exists a unique $\bar{h} : F(A, B, \Box) \to (C, D, \Box)$ such that $\bar{h} \circ i = h$.
- By the general theory, repeatedly applying the free image-total GL-extension to the free rank 0-1 partial GL-algebra gives a bottom-up construction of the free algebra:

$$F_{GL} = \operatorname{colim}_n F^n(A_1, A_0, \Box_0)$$

Free image-total extension

- Let $(A, B, \Box)$ be a partial modal algebra. We say that a homomorphism $i : (A, B, \Box) \to F(A, B, \Box)$ into a partial GL-algebra $F(A, B, \Box)$ is a free image-total GL-extension if for any homomorphism $h : (A, B, \Box) \to (C, D, \Box)$ into a partial GL-algebra such that $h(B) \subseteq D$, there exists a unique $\bar{h} : F(A, B, \Box) \to (C, D, \Box)$ such that $\bar{h} \circ i = h$.
- By the general theory, repeatedly applying the free image-total GL-extension to the free rank 0-1 partial GL-algebra gives a bottom-up construction of the free algebra:

$$F_{GL} = \mathrm{colim}_n \, F^n(A_1, A_0, \Box_0)$$

- Question. What is $F(A, B, \Box)$, concretely?

Free image-total extension

- Let $(A, B, \Box)$ be a partial modal algebra. We say that a homomorphism $i : (A, B, \Box) \to F(A, B, \Box)$ into a partial GL-algebra $F(A, B, \Box)$ is a free image-total GL-extension if for any homomorphism $h : (A, B, \Box) \to (C, D, \Box)$ into a partial GL-algebra such that $h(B) \subseteq D$, there exists a unique $\bar{h} : F(A, B, \Box) \to (C, D, \Box)$ such that $\bar{h} \circ i = h$.
- By the general theory, repeatedly applying the free image-total GL-extension to the free rank 0-1 partial GL-algebra gives a bottom-up construction of the free algebra:

$$F_{GL} = \mathrm{colim}_n \, F^n(A_1, A_0, \Box_0)$$

- Question. What is $F(A, B, \Box)$, concretely?
- Use: frame semantics for partial modal algebras.

Duality for partial GL-algebras

- A partial / one-step frame is a tuple $(X, \sim, R)$, where X is a set, $\sim$ is an equivalence relation on X, and R is a relation such that, for any $x, y \in X$, if xRy and $y \sim y'$ then xRy'.

Duality for partial GL-algebras

- A partial / one-step frame is a tuple $(X, \sim, R)$, where X is a set, $\sim$ is an equivalence relation on X, and R is a relation such that, for any $x, y \in X$, if xRy and $y \sim y'$ then xRy'.
- A partial frame yields a partial modal algebra

$$(X, \sim, R)^+ := (\mathcal{P}(X), \mathcal{P}_\sim(X), \Box_R)$$

and, hence, a 'one-step semantics'.

Duality for partial GL-algebras

- A partial / one-step frame is a tuple $(X, \sim, R)$, where X is a set, $\sim$ is an equivalence relation on X, and R is a relation such that, for any $x, y \in X$, if xRy and $y \sim y'$ then xRy'.
- A partial frame yields a partial modal algebra

$$(X, \sim, R)^+ := (\mathcal{P}(X), \mathcal{P}_\sim(X), \Box_R)$$

and, hence, a 'one-step semantics'.

Proposition

Let $(X, \sim, R)$ be a finite partial frame and $(A, B, \Box)$ its dual partial modal algebra. The following are equivalent:

Duality for partial GL-algebras

- A partial / one-step frame is a tuple $(X, \sim, R)$, where X is a set, $\sim$ is an equivalence relation on X, and R is a relation such that, for any $x, y \in X$, if xRy and $y \sim y'$ then xRy'.
- A partial frame yields a partial modal algebra

$$(X, \sim, R)^+ := (\mathcal{P}(X), \mathcal{P}_\sim(X), \Box_R)$$

and, hence, a 'one-step semantics'.

Proposition

Let $(X, \sim, R)$ be a finite partial frame and $(A, B, \Box)$ its dual partial modal algebra. The following are equivalent:

1. $(A, B, \Box)$ *is a partial GL-algebra,*

Duality for partial GL-algebras

- A partial / one-step frame is a tuple $(X, \sim, R)$, where X is a set, $\sim$ is an equivalence relation on X, and R is a relation such that, for any $x, y \in X$, if xRy and $y \sim y'$ then xRy'.
- A partial frame yields a partial modal algebra

$$(X, \sim, R)^+ := (\mathcal{P}(X), \mathcal{P}_\sim(X), \Box_R)$$

and, hence, a 'one-step semantics'.

Proposition

Let $(X, \sim, R)$ be a finite partial frame and $(A, B, \Box)$ its dual partial modal algebra. The following are equivalent:

1. *$(A, B, \Box)$ is a partial GL-algebra,*
2. *$(X, \sim, R)$ is a partial GL-frame, i.e., for any $x, y \in X$, if xRy then there exists $y' \sim y$ such that $R[y'] \subsetneq R[y]$.*

Duality for partial GL-algebras

- A partial / one-step frame is a tuple $(X, \sim, R)$, where X is a set, $\sim$ is an equivalence relation on X, and R is a relation such that, for any $x, y \in X$, if xRy and $y \sim y'$ then xRy'.
- A partial frame yields a partial modal algebra

$$(X, \sim, R)^+ := (\mathcal{P}(X), \mathcal{P}_\sim(X), \Box_R)$$

and, hence, a 'one-step semantics'.

Proposition

Let $(X, \sim, R)$ be a finite partial frame and $(A, B, \Box)$ its dual partial modal algebra. The following are equivalent:

1. *$(A, B, \Box)$ is a partial GL-algebra,*
2. *$(X, \sim, R)$ is a partial GL-frame, i.e., for any $x, y \in X$, if xRy then there exists $y' \sim y$ such that $R[y'] \subsetneq R[y]$.*

- NB: *partial* GL-frames may contain reflexive points.

Free image-total extension, on frames

- Let $(A, B, \Box)$ be a finite partial modal algebra with dual frame $(X, \sim, R)$.

Free image-total extension, on frames

- Let $(A, B, \Box)$ be a finite partial modal algebra with dual frame $(X, \sim, R)$.
- Goal: Describe the dual frame of $F(A, B, \Box)$.

Free image-total extension, on frames

- Let $(A, B, \Box)$ be a finite partial modal algebra with dual frame $(X, \sim, R)$.
- Goal: Describe the dual frame of $F(A, B, \Box)$.
- The free image-total GL-extension $F(A, B, \Box)$ is a quotient of a coproduct $A + V(A)$.

Free image-total extension, on frames

- Let $(A, B, \Box)$ be a finite partial modal algebra with dual frame $(X, \sim, R)$.
- Goal: Describe the dual frame of $F(A, B, \Box)$.
- The free image-total GL-extension $F(A, B, \Box)$ is a quotient of a coproduct $A + V(A)$.
- Hence, the frame dual to $F(A, B, \Box)$ is a subframe of a product $X \times \mathcal{P}(X)$.

Free image-total extension, on frames

- Let $(A, B, \Box)$ be a finite partial modal algebra with dual frame $(X, \sim, R)$.
- Goal: Describe the dual frame of $F(A, B, \Box)$.
- The free image-total GL-extension $F(A, B, \Box)$ is a quotient of a coproduct $A + V(A)$.
- Hence, the frame dual to $F(A, B, \Box)$ is a subframe of a product $X \times \mathcal{P}(X)$.
- Which subframe?

GL-suitable points

- Let $(X, \sim, R)$ be a finite partial frame.

GL-suitable points

- Let $(X, \sim, R)$ be a finite partial frame.
- We say a pair $(x, T) \in X \times \mathcal{P}(X)$ is GL-suitable if

GL-suitable points

- Let $(X, \sim, R)$ be a finite partial frame.
- We say a pair $(x, T) \in X \times \mathcal{P}(X)$ is GL-suitable if
  1. for any $y \in X$, xRy iff there exists $t \in T$ such that $y \sim t$,

GL-suitable points

- Let $(X, \sim, R)$ be a finite partial frame.
- We say a pair $(x, T) \in X \times \mathcal{P}(X)$ is GL-suitable if
  1. for any $y \in X$, xRy iff there exists $t \in T$ such that $y \sim t$,
  2. for any $y \in T$, there exists $S \subsetneq T$ such that (y, S) is GL-suitable.

GL-suitable points

- Let $(X, \sim, R)$ be a finite partial frame.
- We say a pair $(x, T) \in X \times \mathcal{P}(X)$ is GL-suitable if
  1. for any $y \in X$, xRy iff there exists $t \in T$ such that $y \sim t$,
  2. for any $y \in T$, there exists $S \subsetneq T$ such that (y, S) is GL-suitable.
- This definition looks circular, but ...

GL-suitable points

- Let $(X, \sim, R)$ be a finite partial frame.
- We say a pair $(x, T) \in X \times \mathcal{P}(X)$ is GL-suitable if
  1. for any $y \in X$, xRy iff there exists $t \in T$ such that $y \sim t$,
  2. for any $y \in T$, there exists $S \subsetneq T$ such that (y, S) is GL-suitable.
- This definition looks circular, but ...
- ... deciding whether (x, T) is GL-suitable only depends on whether (y, S) is GL-suitable for S with $|S| < |T|$.

GL-suitable points

- Let $(X, \sim, R)$ be a finite partial frame.
- We say a pair $(x, T) \in X \times \mathcal{P}(X)$ is GL-suitable if
  1. for any $y \in X$, xRy iff there exists $t \in T$ such that $y \sim t$,
  2. for any $y \in T$, there exists $S \subsetneq T$ such that (y, S) is GL-suitable.
- This definition looks circular, but ...
- ... deciding whether (x, T) is GL-suitable only depends on whether (y, S) is GL-suitable for S with $|S| < |T|$.
- In particular, the definition can be unravelled as follows:

GL-suitable points

- Let $(X, \sim, R)$ be a finite partial frame.
- We say a pair $(x, T) \in X \times \mathcal{P}(X)$ is GL-suitable if
  1. for any $y \in X$, xRy iff there exists $t \in T$ such that $y \sim t$,
  2. for any $y \in T$, there exists $S \subsetneq T$ such that (y, S) is GL-suitable.
- This definition looks circular, but ...
- ... deciding whether (x, T) is GL-suitable only depends on whether (y, S) is GL-suitable for S with $|S| < |T|$.
- In particular, the definition can be unravelled as follows:
  - A pair of the form $(x, \emptyset)$ is GL-suitable if $R[x] = \emptyset$;

GL-suitable points

- Let $(X, \sim, R)$ be a finite partial frame.
- We say a pair $(x, T) \in X \times \mathcal{P}(X)$ is GL-suitable if
  1. for any $y \in X$, xRy iff there exists $t \in T$ such that $y \sim t$,
  2. for any $y \in T$, there exists $S \subsetneq T$ such that (y, S) is GL-suitable.
- This definition looks circular, but ...
- ... deciding whether (x, T) is GL-suitable only depends on whether (y, S) is GL-suitable for S with $|S| < |T|$.
- In particular, the definition can be unravelled as follows:
  - A pair of the form $(x, \emptyset)$ is GL-suitable if $R[x] = \emptyset$;
  - A pair of the form $(x, \{y\})$ is GL-suitable if $R[x] = [y]_\sim$ and $R[y] = \emptyset$;

GL-suitable points

- Let $(X, \sim, R)$ be a finite partial frame.
- We say a pair $(x, T) \in X \times \mathcal{P}(X)$ is GL-suitable if
  1. for any $y \in X$, xRy iff there exists $t \in T$ such that $y \sim t$,
  2. for any $y \in T$, there exists $S \subsetneq T$ such that (y, S) is GL-suitable.
- This definition looks circular, but ...
- ... deciding whether (x, T) is GL-suitable only depends on whether (y, S) is GL-suitable for S with $|S| < |T|$.
- In particular, the definition can be unravelled as follows:
  - A pair of the form $(x, \emptyset)$ is GL-suitable if $R[x] = \emptyset$;
  - A pair of the form $(x, \{y\})$ is GL-suitable if $R[x] = [y]_\sim$ and $R[y] = \emptyset$;
  - (...)

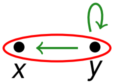
GL-suitable points

- Let $(X, \sim, R)$ be a finite partial frame.
- We say a pair $(x, T) \in X \times \mathcal{P}(X)$ is GL-suitable if
  1. for any $y \in X$, xRy iff there exists $t \in T$ such that $y \sim t$,
  2. for any $y \in T$, there exists $S \subsetneq T$ such that (y, S) is GL-suitable.
- This definition looks circular, but ...
- ... deciding whether (x, T) is GL-suitable only depends on whether (y, S) is GL-suitable for S with $|S| < |T|$.
- In particular, the definition can be unravelled as follows:
  - A pair of the form $(x, \emptyset)$ is GL-suitable if $R[x] = \emptyset$;
  - A pair of the form $(x, \{y\})$ is GL-suitable if $R[x] = [y]_\sim$ and $R[y] = \emptyset$;
  - (...)
- $G(X, \sim, R) := \{(x, T) \in X \times \mathcal{P}(X) \mid (x, T) \text{ is GL-suitable}\}$.

GL-suitable points

Example

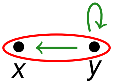
$(X, \sim, R)$



GL-suitable points

Example

$(X, \sim, R)$ $\qquad\qquad$ $X \times \mathcal{P}(X)$



$(x, \emptyset)$ • $\qquad$ • $(y, \emptyset)$

$(x, \{x\})$ • $\qquad$ • $(y, \{x\})$

$(x, \{y\})$ • $\qquad$ • $(y, \{y\})$

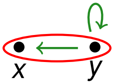
$(x, \{x, y\})$ • $\qquad$ • $(y, \{x, y\})$

- A pair $(x, T) \in X \times \mathcal{P}(X)$ is GL-suitable if

GL-suitable points

Example

$(X, \sim, R)$ $\qquad\qquad$ $X \times \mathcal{P}(X)$



$(x, \emptyset) \bullet \qquad \bullet (y, \emptyset)$

$(x, \{x\}) \bullet \qquad \bullet (y, \{x\})$

$(x, \{y\}) \bullet \qquad \bullet (y, \{y\})$

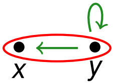
$(x, \{x, y\}) \bullet \qquad \bullet (y, \{x, y\})$

- A pair $(x, T) \in X \times \mathcal{P}(X)$ is GL-suitable if
  1. $R[x] = [T]_\sim$,

GL-suitable points

Example

$(X, \sim, R)$



$X \times \mathcal{P}(X)$

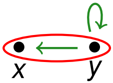
$(x, \emptyset)$ •	• ~~$(y, \emptyset)$~~
~~$(x, \{x\})$~~ •	• $(y, \{x\})$
~~$(x, \{y\})$~~ •	• $(y, \{y\})$
~~$(x, \{x, y\})$~~ •	• $(y, \{x, y\})$

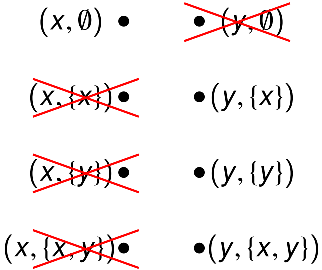
- A pair $(x, T) \in X \times \mathcal{P}(X)$ is GL-suitable if
  1. $R[x] = [T]_{\sim}$,

GL-suitable points

Example

$(X, \sim, R)$ $\quad\quad$ $X \times \mathcal{P}(X)$



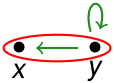


- A pair $(x, T) \in X \times \mathcal{P}(X)$ is GL-suitable if
  1. $R[x] = [T]_{\sim}$,
  2. for any $y \in T$, there exists $S \subsetneq T$ such that (y, S) is GL-suitable.

GL-suitable points

Example

$(X, \sim, R)$



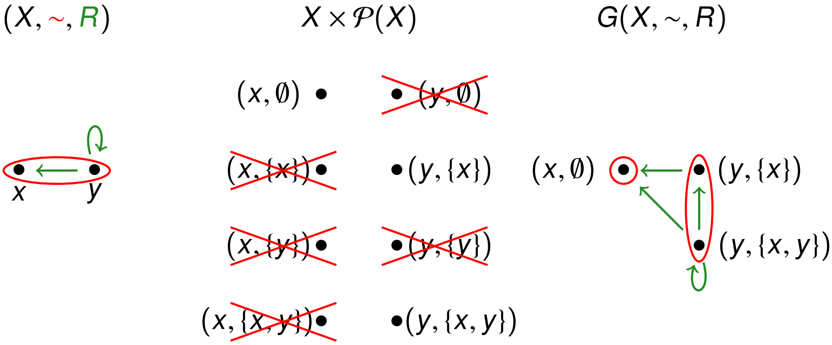
$X \times \mathcal{P}(X)$

$(x, \emptyset)$	~~$(y, \emptyset)$~~
~~$(x, \{x\})$~~	$(y, \{x\})$
~~$(x, \{y\})$~~	~~$(y, \{y\})$~~
$(x,$ ~~$\{x, y\})$~~	$(y, \{x, y\})$

- A pair $(x, T) \in X \times \mathcal{P}(X)$ is GL-suitable if
  1. $R[x] = [T]_{\sim}$,
  2. for any $y \in T$, there exists $S \subsetneq T$ such that (y, S) is GL-suitable.

GL-suitable points

Example



- A pair $(x, T) \in X \times \mathcal{P}(X)$ is GL-suitable if
  1. $R[x] = [T]_{\sim}$,
  2. for any $y \in T$, there exists $S \subsetneq T$ such that (y, S) is GL-suitable.

GL-suitable points give extension

Proposition

Let $(A, B, \Box)$ be a partial modal algebra with dual partial frame $(X, \sim, R)$. The partial frame $G(X, \sim, R)$ of GL-suitable points in $X \times \mathcal{P}(X)$ is dual to the free image-total GL-extension $F(A, B, \Box)$:

$$F(A, B, \Box) = (G(X, \sim, R))^+.$$

GL-suitable points give extension

Proposition

Let $(A, B, \Box)$ *be a partial modal algebra with dual partial frame* $(X, \sim, R)$*. The partial frame* $G(X, \sim, R)$ *of GL-suitable points in* $X \times \mathcal{P}(X)$ *is dual to the free image-total GL-extension* $F(A, B, \Box)$*:*

$$F(A, B, \Box) = (G(X, \sim, R))^{+}.$$

Proof.

Use the definition of GL-suitability and duality. □

Free GL-algebras

- Let $\{p_1, \ldots, p_k\}$ be propositional variables.

Free GL-algebras

- Let $\{p_1, \ldots, p_k\}$ be propositional variables.
- Let $(A_1, A_0, \Box_0)$ be the free rank 0-1 partial GL-algebra on the k propositional variables.

Free GL-algebras

- Let $\{p_1, \ldots, p_k\}$ be propositional variables.
- Let $(A_1, A_0, \Box_0)$ be the free rank 0-1 partial GL-algebra on the k propositional variables.
- Let $(A_{n+1}, A_n, \Box_n) := F^n(A_1, A_0, \Box_0)$, which is equal to $(G^n(X_1, \sim_1, R_1))^+$.

Free GL-algebras

- Let $\{p_1, \ldots, p_k\}$ be propositional variables.
- Let $(A_1, A_0, \Box_0)$ be the free rank 0-1 partial GL-algebra on the k propositional variables.
- Let $(A_{n+1}, A_n, \Box_n) := F^n(A_1, A_0, \Box_0)$, which is equal to $(G^n(X_1, \sim_1, R_1))^+$.

Theorem

Free GL-algebras

- Let $\{p_1, \ldots, p_k\}$ be propositional variables.
- Let $(A_1, A_0, \Box_0)$ be the free rank 0-1 partial GL-algebra on the k propositional variables.
- Let $(A_{n+1}, A_n, \Box_n) := F^n(A_1, A_0, \Box_0)$, which is equal to $(G^n(X_1, \sim_1, R_1))^+$.

Theorem

1. $(A_{n+1}, A_n, \Box_n)$ *is the partial GL-algebra of GL-equivalence classes modal formulas in k variables of rank n and $n+1$.*

Free GL-algebras

- Let $\{p_1, \ldots, p_k\}$ be propositional variables.
- Let $(A_1, A_0, \Box_0)$ be the free rank 0-1 partial GL-algebra on the k propositional variables.
- Let $(A_{n+1}, A_n, \Box_n) := F^n(A_1, A_0, \Box_0)$, which is equal to $(G^n(X_1, \sim_1, R_1))^+$.

Theorem

1. *$(A_{n+1}, A_n, \Box_n)$ is the partial GL-algebra of GL-equivalence classes modal formulas in k variables of rank n and $n+1$.*
2. *The free GL-algebra on k variables is the colimit (= union) of the chain $A_0 \hookrightarrow A_1 \hookrightarrow A_2 \hookrightarrow \ldots$ of Boolean algebras, with $\Box$ defined in the only possible way.*

Discussion

- The relationship to normal forms for GL?

Discussion

- The relationship to normal forms for GL?
- One-step frames and filtration?

Discussion

- The relationship to normal forms for GL?
- One-step frames and filtration?
- One-step frames and bi-modal logic?

Discussion

- The relationship to normal forms for GL?
- One-step frames and filtration?
- One-step frames and bi-modal logic?
- The role of fix-points in finding the ‘suitable points’?

Discussion

- The relationship to normal forms for GL?
- One-step frames and filtration?
- One-step frames and bi-modal logic?
- The role of fix-points in finding the ‘suitable points’?
- What makes one flattened rule-version of the GL-axiom:

$$b \leq \Box a \to a \quad \Rightarrow \quad \Box b \leq \Box a$$

‘better’ than the other one?

$$\Box a \leq b \quad \Rightarrow \quad \Box(b \to a) \leq \Box a$$

Discussion

- The relationship to normal forms for GL?
- One-step frames and filtration?
- One-step frames and bi-modal logic?
- The role of fix-points in finding the ‘suitable points’?
- What makes one flattened rule-version of the GL-axiom:

$$b \leq \Box a \to a \quad \Rightarrow \quad \Box b \leq \Box a$$

  ‘better’ than the other one?

$$\Box a \leq b \quad \Rightarrow \quad \Box(b \to a) \leq \Box a$$

- Your questions / answers?