Stone duality in ten minutes

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LIAFA, Journée des Entrants

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- In these ten minutes: an example of Stone duality in action.

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- Solution 2: Apply Stone duality! $\leftarrow$ will explain this now

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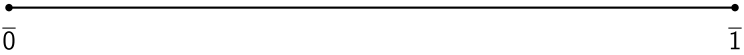

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Basic open set, for $(a_1, \dots, a_n) \in \{0,1\}^n$:

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The sets $K_{(a_1,\ldots,a_n)}$ are clopen, and they form a basis!

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- Let B be the collection of all clopen subsets of the Cantor space X.

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- Then B forms a Boolean algebra: it contains $\emptyset$, X, and is closed under the operations $\cup$, $\cap$ and $()^c$.

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E.g., the clopen set $K_{(0,1)}$ corresponds to the class of the formula $\neg p_0 \wedge p_1$.

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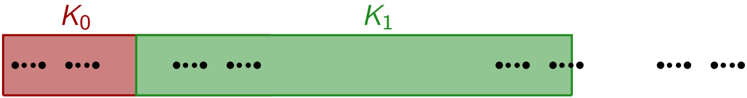

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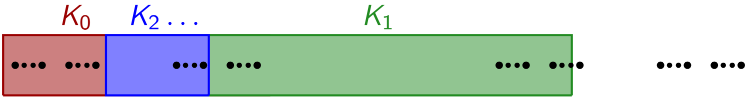

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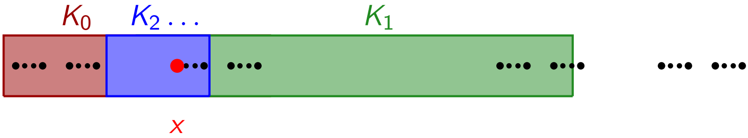

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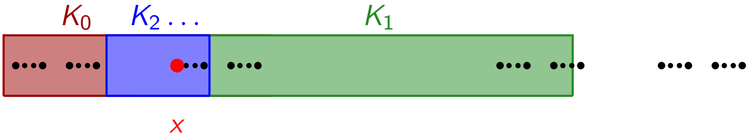

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- Since Cantor space is compact, there exists a point $x \in \bigcap_{n=0}^{\infty} K_n$;
- This point x is a model of all of $\{\phi_0, \phi_1, \dots\}$.

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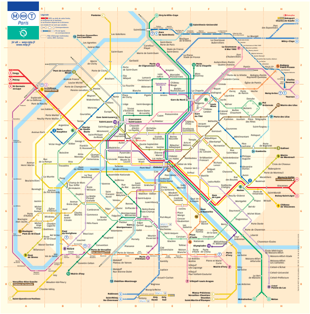

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- Combinatorial solution can be used to answer question algorithmically;
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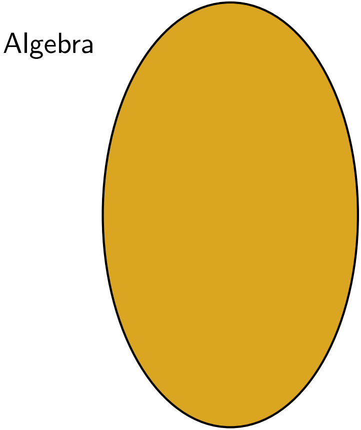
- In other cases, where the combinatorial structure that is needed to solve a problem is not obvious from the start...

- Stone duality may be used as a guide for finding it.

The general picture



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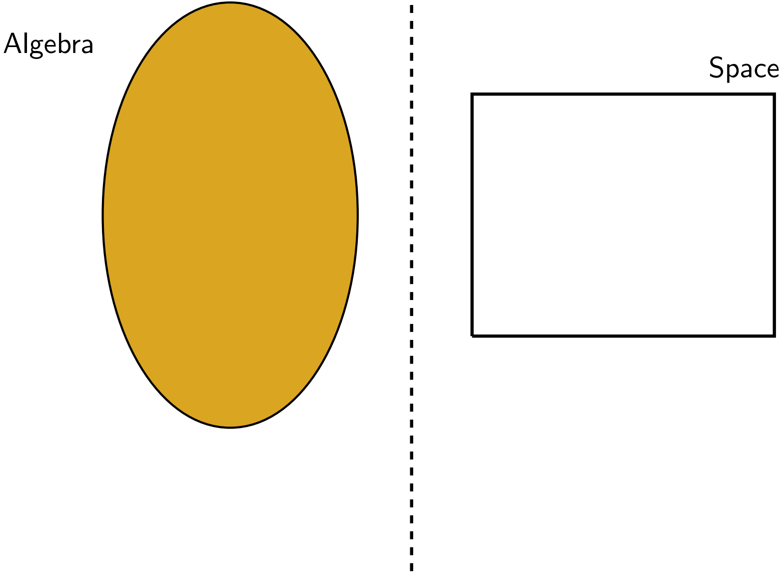

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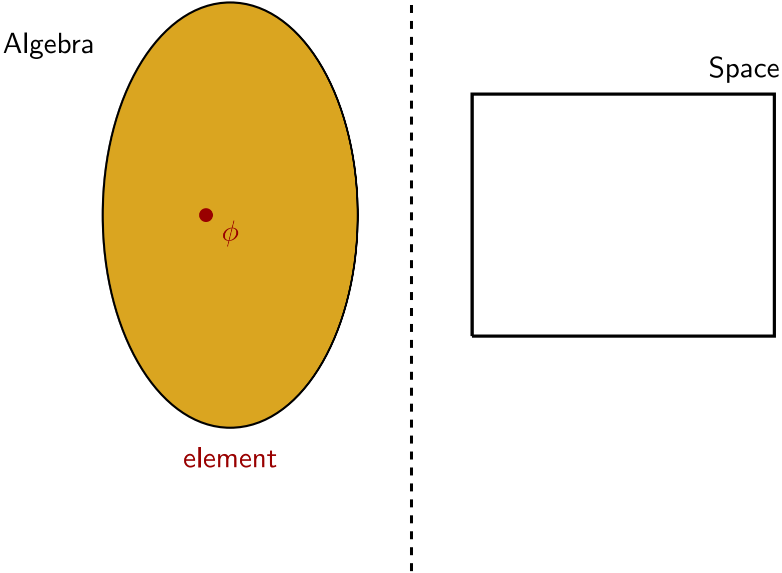

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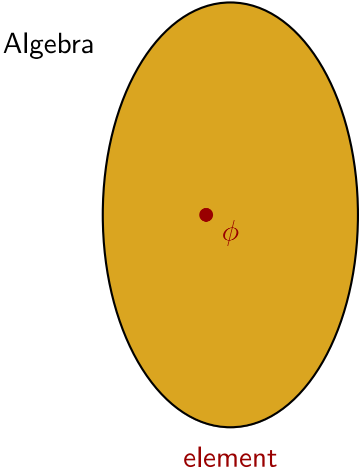

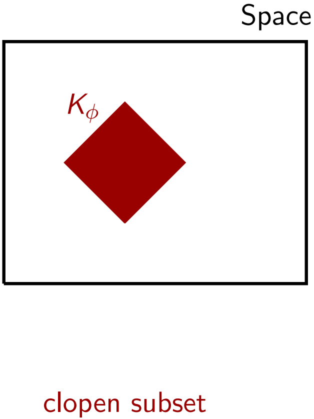

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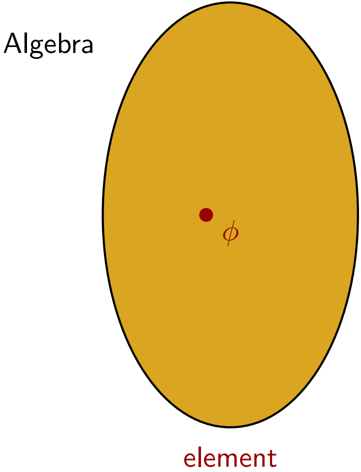

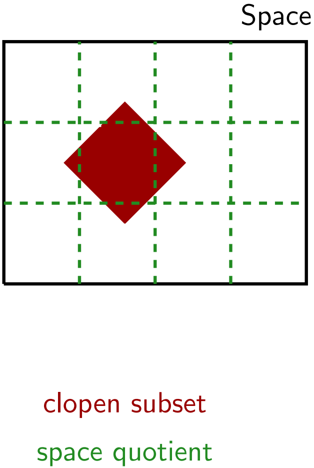

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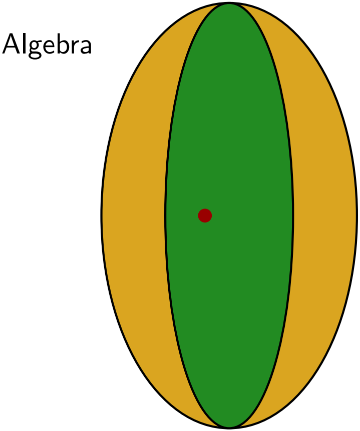
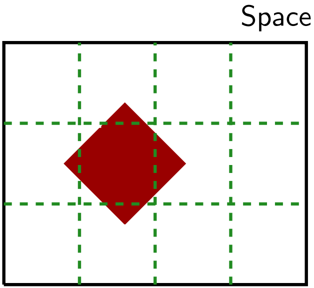


element

subalgebra



clopen subset

space quotient

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- Applications to language theory: Generalizing the recent work of Gehrke, Gregorieff, Pin on Stone duality for classical automata to cost automata, languages defined by logics, ...

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