# A non-commutative Priestley duality

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  1. Gödel's Completeness Theorem for FO logic via Stone duality;
  2. A new duality for skew distributive lattices.

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# Prelude: Logic via Algebra

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  - E.g., think of: Hilbert-style classical propositional logic, Gentzen-style intuitionistic first-order logic, . . .
- Often, interderivability, $\vdash\dashv$, is an equivalence relation on formulas;
- The collection of $\vdash$-equivalence classes of formulas is called the Lindenbaum algebra for the logic.

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- Properties of the logic translate to properties of the algebra;
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- Specific examples that you may encounter during this meeting:
  Heyting algebras to study intuitionistic logic;
  study of distributive lattices of formal languages;
  De Vries algebras to study modal $\mu$-calculus;
  triposes to study higher-order logic
  (add your own example to this list) . . .

# Prelude: Logic via Algebra

## A plethora of ordered algebras

| Logic | Algebras |
|---|---|
| Classical Propositional Logic | Boolean algebras |
| Intuitionistic Propositional Logic | Heyting algebras |
| Modal Logic **K** | Modal algebras |
| Positive Propositional Logic | Distributive lattices |

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### Theorem (Gödel 1930)

*If a sentence $\psi$ is not provable from a set of sentences $\Gamma$ in classical first-order logic, then there exists a model in which each sentence in $\Gamma$ holds, and in which $\psi$ does not hold.*

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- Method: use the Lindenbaum algebra, Stone duality, and an appropriate theorem from general topology.

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- Morally: propositions are represented as sets of 'possible worlds' in the dual space $X$.
- In particular, $\widehat{\varphi} \not\subseteq \widehat{\psi}$, so we find a point $\mathfrak{p} \in \widehat{\varphi} \setminus \widehat{\psi}$.

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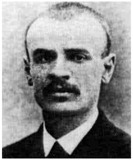
Theorem (Baire)



*If $X$ is a compact Hausdorff topological space, then a countable intersection of dense open sets is dense.*

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- Using Baire, pick a special point $\mathfrak{p}$ in $\widehat{\varphi} \cap \widehat{\psi}^c$.
- By the (full) Truth Lemma, the associated model $M_{\mathfrak{p}}$ makes $\varphi$ true and $\psi$ false. □

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- Rest of this talk: modifications of the first kind.

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**Theorem (Priestley 1970)**

*For any distributive lattice $D$, there exists a compact ordered topological space $X$ and an isomorphism from $D$ onto a basis of clopen downsets of $X$.*

- Morally: in positive logic, propositions are represented as downward closed subsets of possible worlds in dual space $X$.

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(but the idea might be transferable to a linguistic/philosophical/quantum example...?)

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- The algorithms may make mistakes sometimes.
- The program thus needs to manipulate partial functions $\mathbb{N} \to \mathbb{N}$, using the operations 'restriction' and 'override':

(cf. Berendsen, Jansen, Schmaltz, and Vaandrager 2010)

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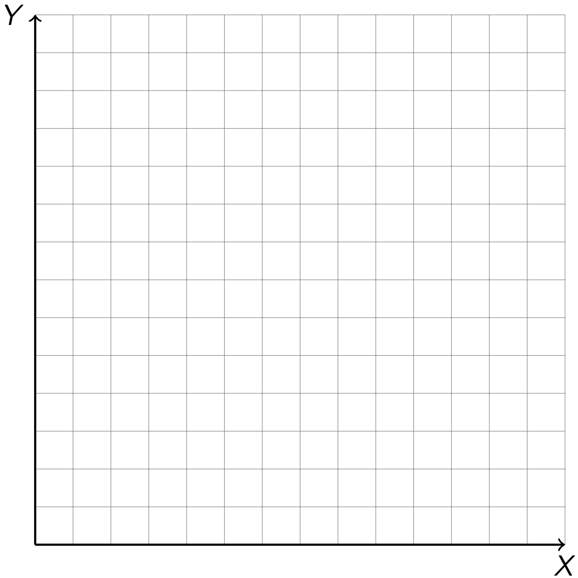
- Distributive skew lattices are the algebraic structures for a positive logic without commutativity.
- The example of the algebra of partial functions plays a role similar to the role of the power set algebra in classical logic.
- Question: is there also a duality for skew lattices?

# Example, pictorially

## Two partial functions



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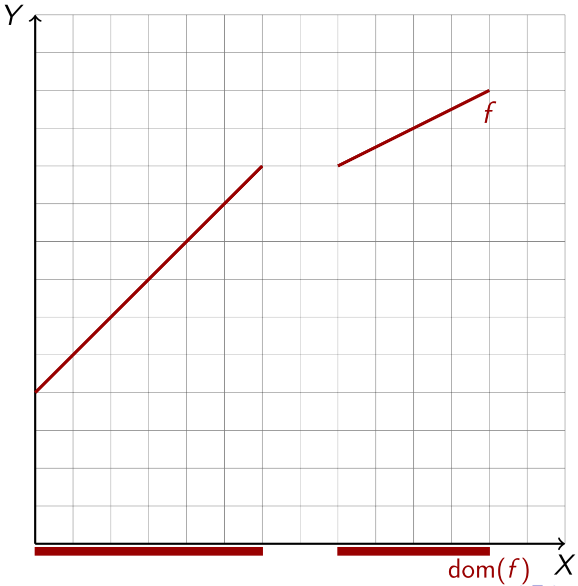

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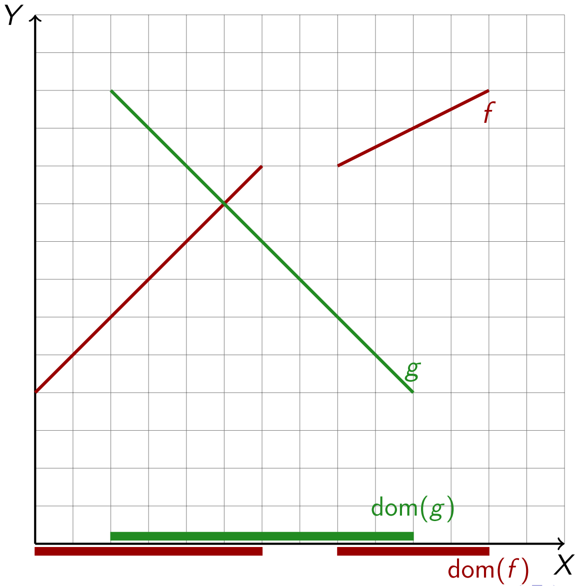

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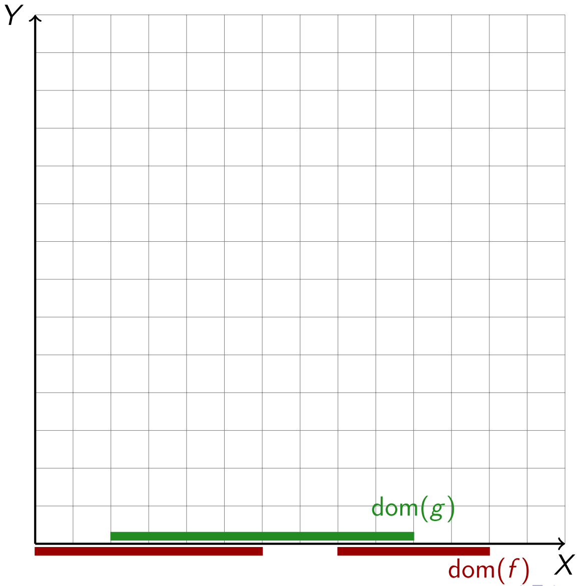

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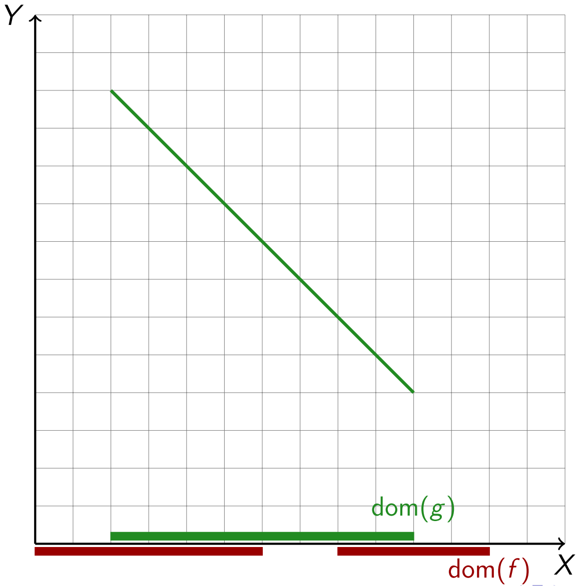

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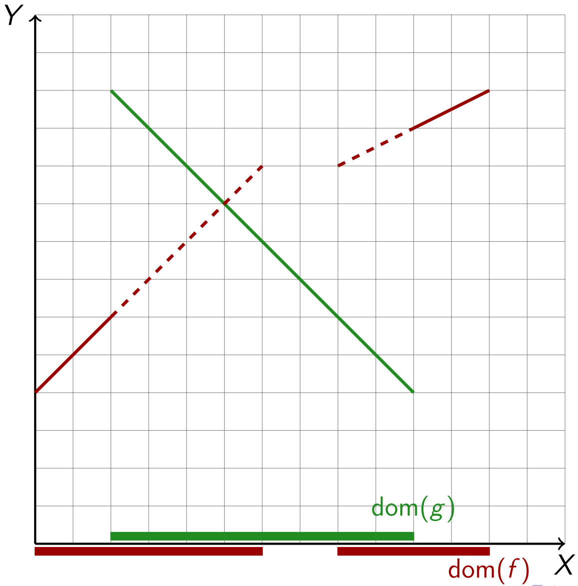

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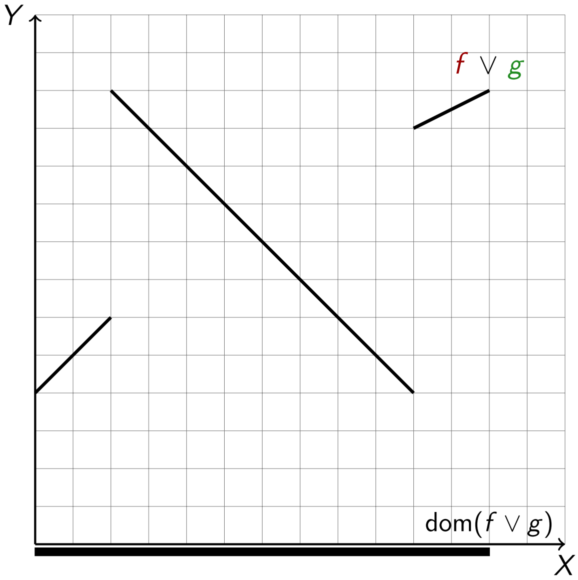

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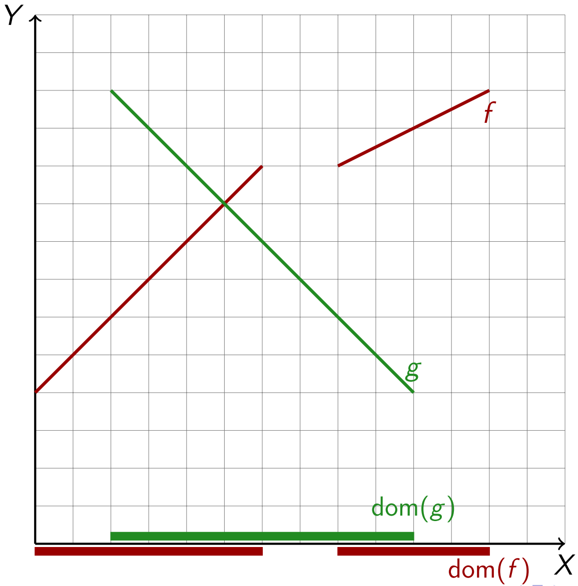

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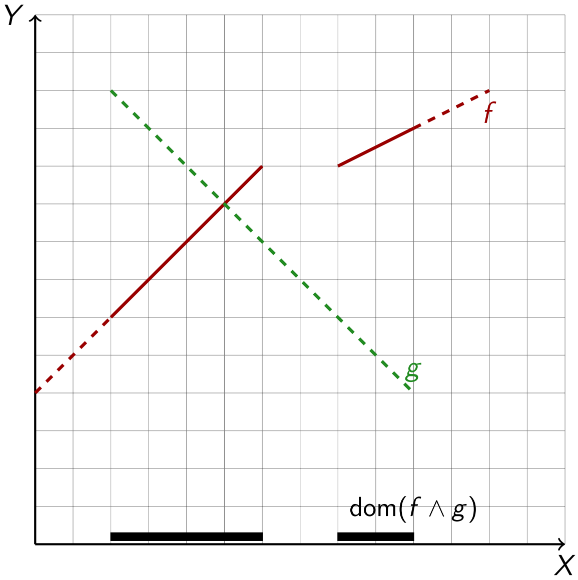

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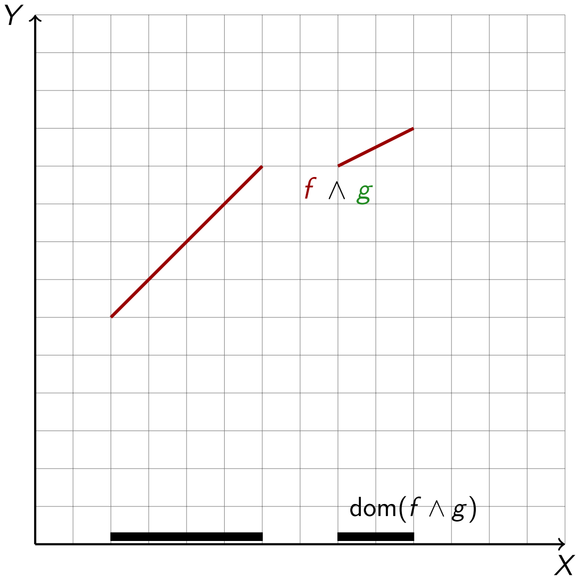

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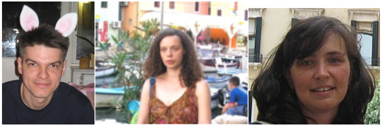
- Answer: 

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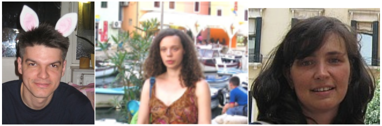
- Answer: 

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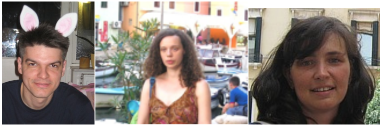
- Answer: 

  (Bauer, Cvetko-Vah, Kudryavtseva, Gehrke, vG 2012)

  Yes!

- Proper setting for this duality: sheaves (see next slide).

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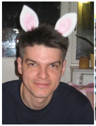
- Answer:   

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*For any distributive skew lattice $S$, there exists a sheaf on a locally compact ordered topological space $X$ and an isomorphism from $S$ onto the local sections over compact-open downsets of $X$.*

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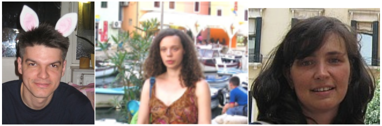
- Answer: 

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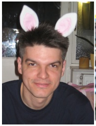
- Answer:   

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- Proof...

  ...is in the paper! (available online)

# Sections of sheaves

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- First part: Stone duality and Baire's theorem $\rightsquigarrow$ Gödel completeness theorem for first-order logic;
- Second part: removing negation and commutativity $\rightsquigarrow$ a duality for distributive skew lattices.

# Further work

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