A non-commutative Priestley duality

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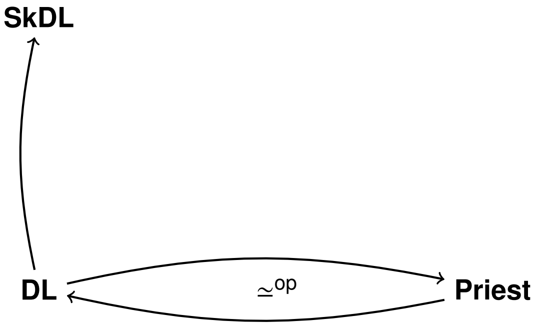

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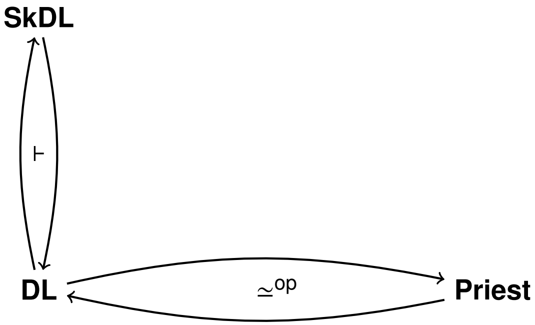

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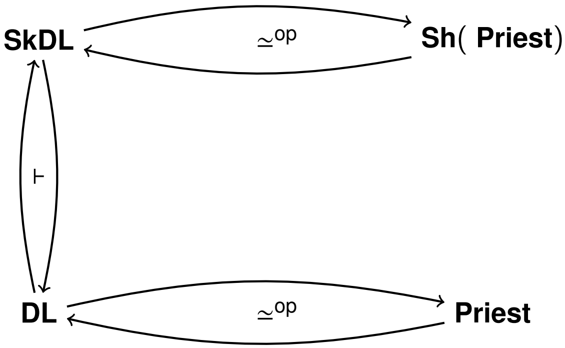

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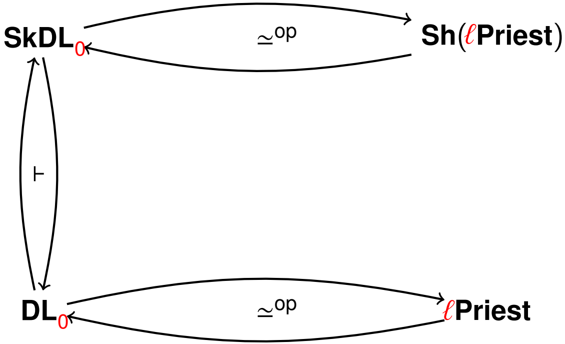

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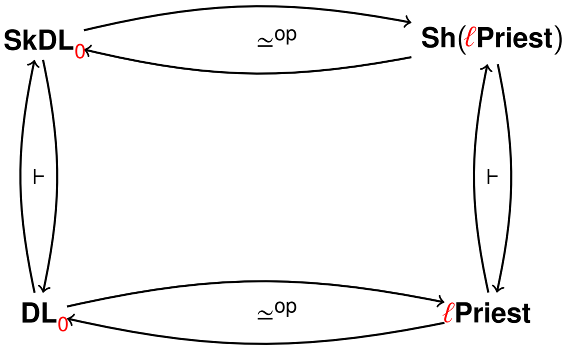

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- We denote the empty function by 0.

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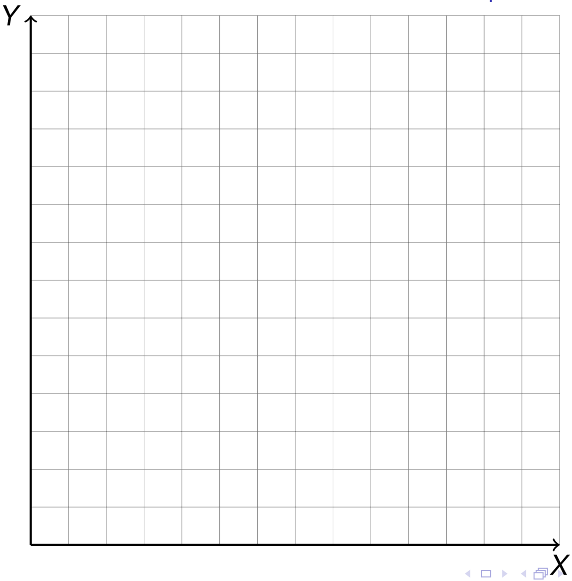
Two partial functions



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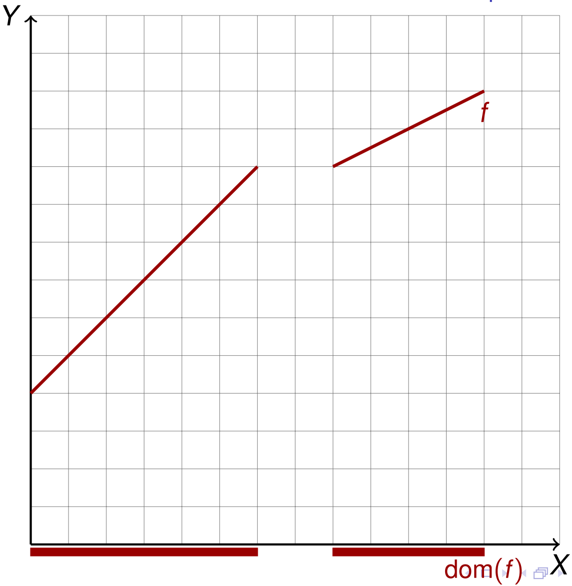

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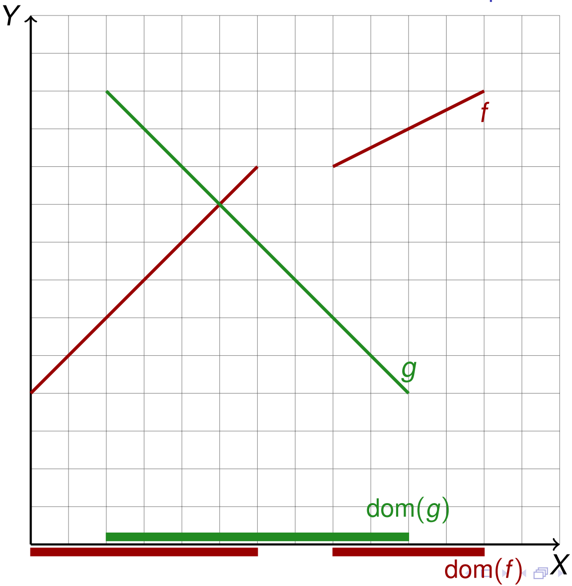

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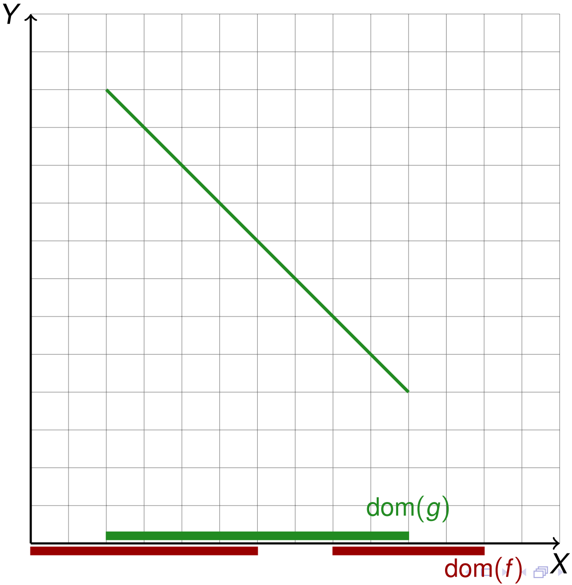

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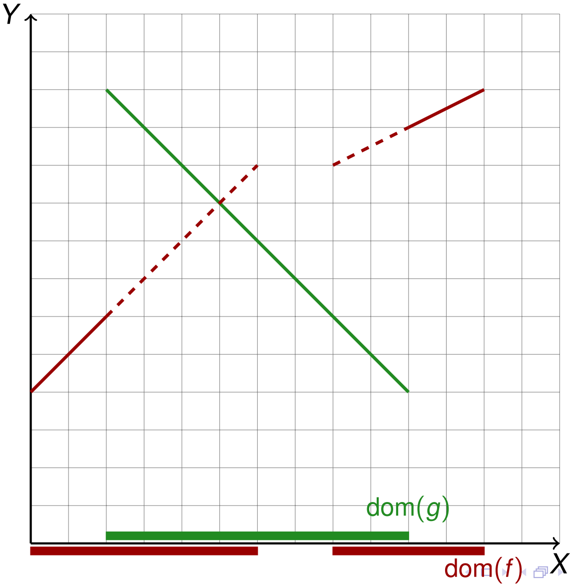

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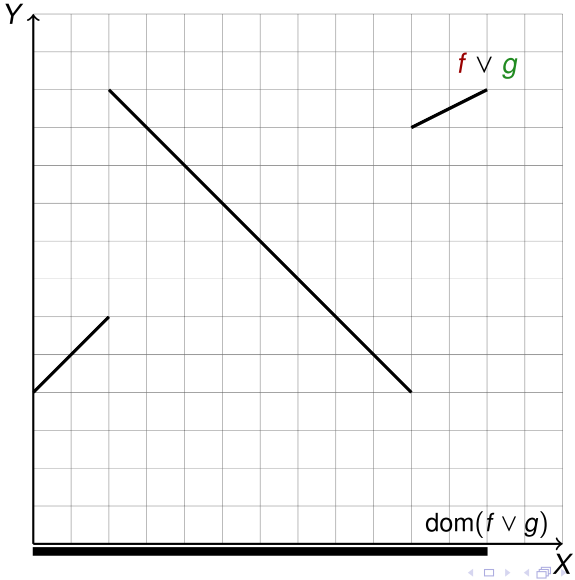

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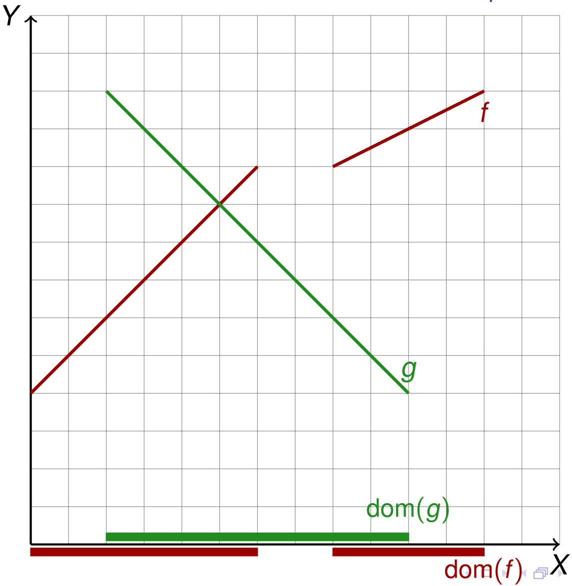

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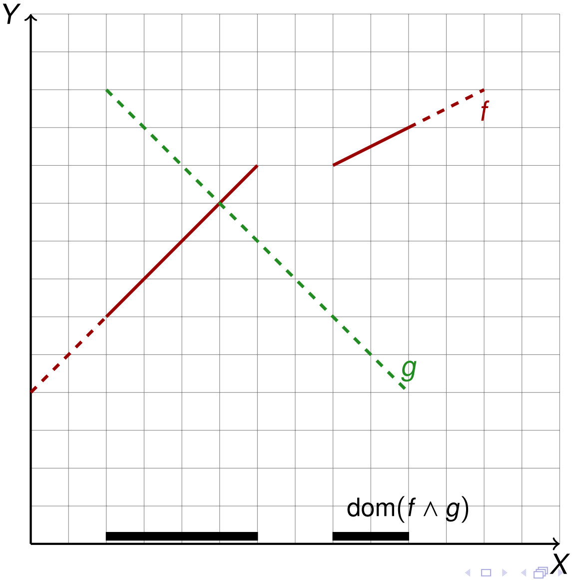

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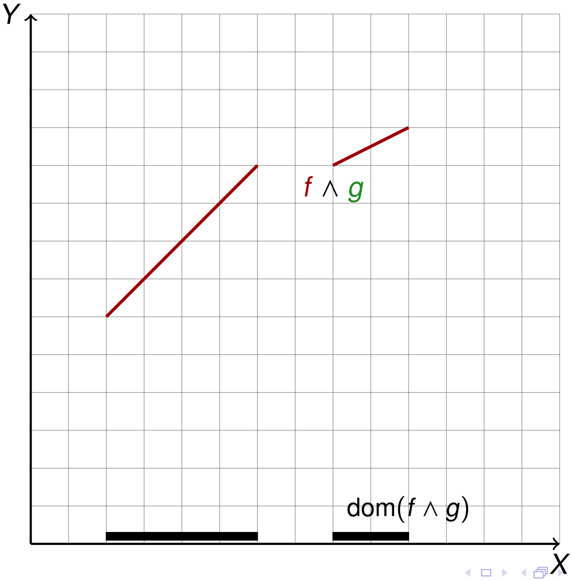

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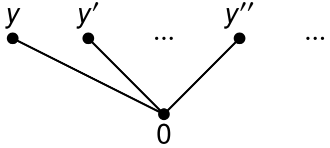
- The operations $\wedge$ and $\vee$ are almost never commutative...
- ... only when the codomain set Y consists of one point.
- When X consists of one point, $\mathcal{P}(X, Y)$ looks simple:



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- Therefore, a homomorphism of skew lattices is a function which preserves 0, $\wedge$ and $\vee$.

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- A skew lattice is distributive when

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Theorem (Leech (1989))

Let S be a skew lattice. The quotient $S/\mathcal{L}$ is the maximal right-handed image of S, $S/\mathcal{R}$ is the maximal left-handed image of S, $S/\mathcal{D}$ is the maximal lattice image of S, and the following diagram is a pullback:

$$\begin{array}{ccc} S & \twoheadrightarrow & S/\mathcal{R} \\ \downarrow & & \downarrow \\ S/\mathcal{L} & \twoheadrightarrow & S/\mathcal{D} \end{array}$$

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- In the case $S \subseteq \mathcal{P}(X, Y)$, what is the lattice $S/\mathcal{D}$?
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- For a skew distributive lattice, $S/\mathcal{D}$ is a distributive lattice.

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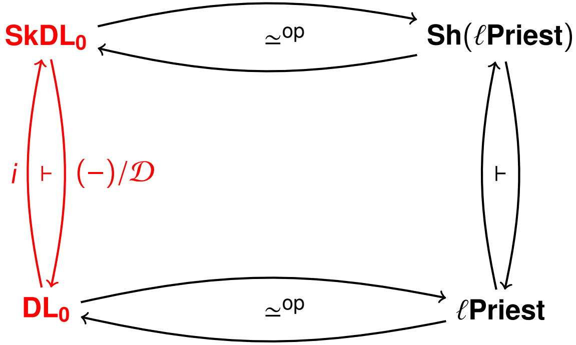

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  - What additional structure does one need on the sets X and Y to be able to recover the skew lattice?

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$\Rightarrow$ Duality with local Priestley spaces: ℓ**Priest**.

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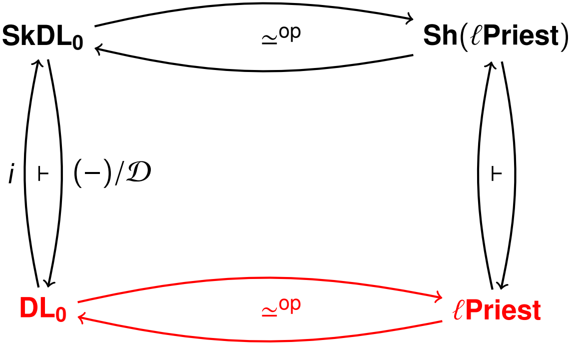

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- In the commutative case, an algebra element corresponds to a subset of the space.
- For a skew lattice S, we can consider the dual space X of the distributive lattice $S/\mathcal{D}$.
- Then X is (isomorphic to) the space of morphisms $S \to \mathbf{2}$.
- An element of S yields a subset of X, but more ...

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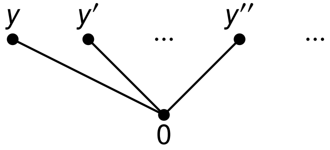

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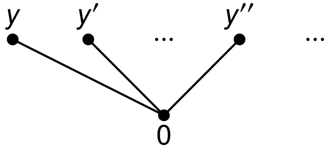

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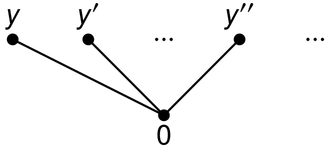

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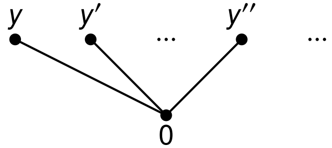

- The quotient $S/\mathcal{D}$ is **2**.
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- For duality, we need all points from the non-zero $\mathcal{D}$-class.

Finding points

Proposition (Largest primitive quotient)

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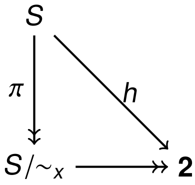

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2. *For any primitive quotient $S \twoheadrightarrow P'$ replacing $S \twoheadrightarrow S/{\sim_x}$ in the above square, there is a unique $t : S/{\sim_x} \to P'$ such that $t \circ \pi = \pi'$.*

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- Indeed, if $S = \mathcal{P}(X, Y)$, then the set underlying the dual space of S is X and $E_x = Y$ for every $x \in X$.
- If S is commutative, then X is the usual dual space of S and $E_x = \{*\}$ for every $x \in X$.

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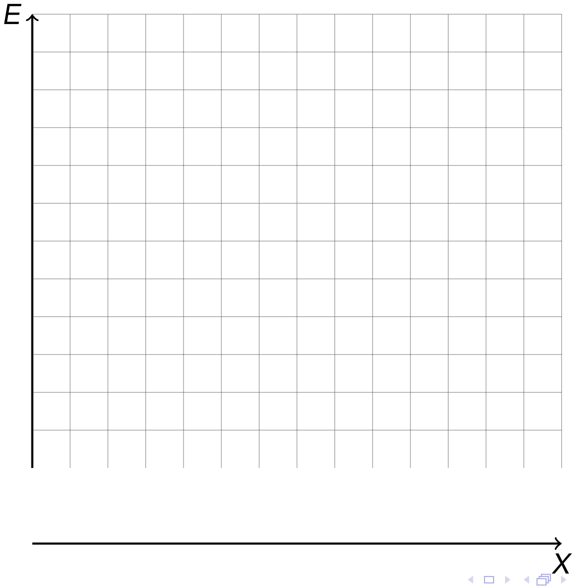
- The topology on E is given by taking the sets $\mathbf{im}(s_a)$ as a subbasis.

Dual object, pictorially



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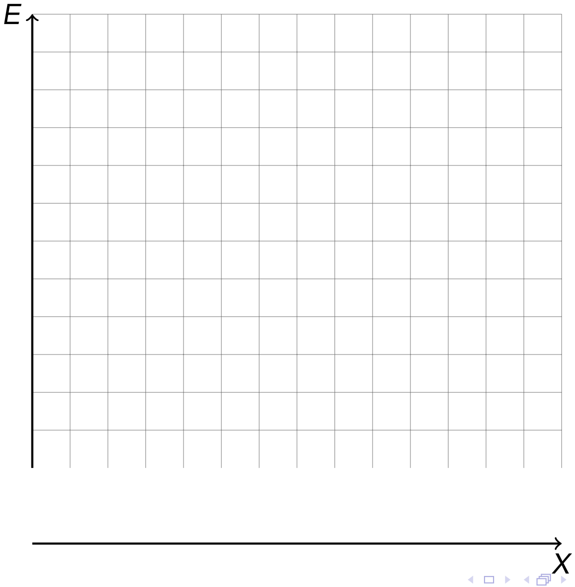

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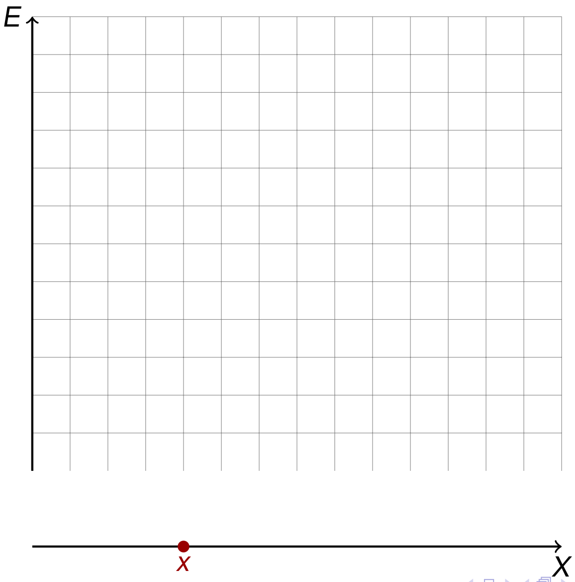

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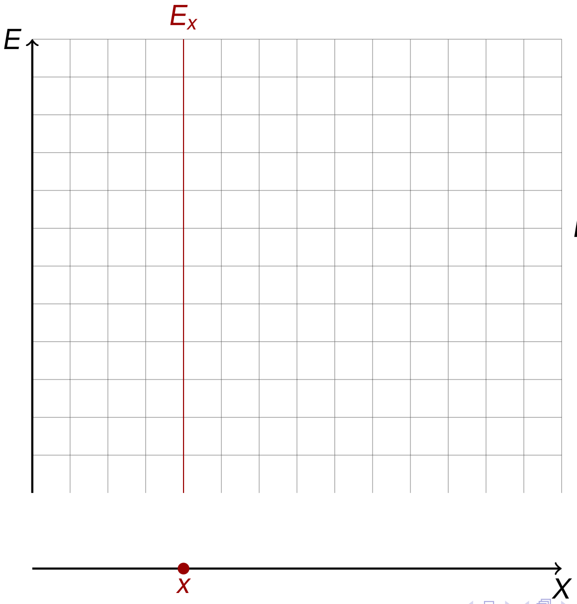

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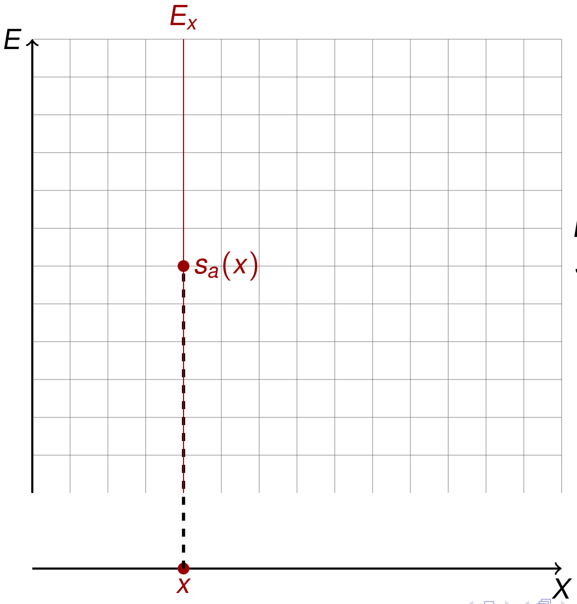

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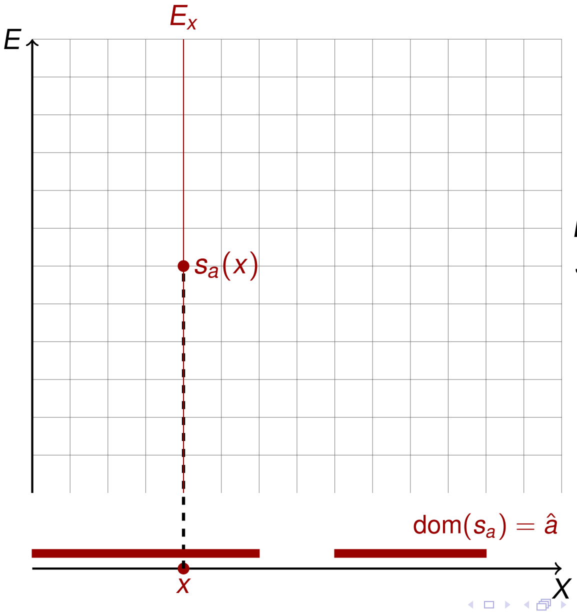

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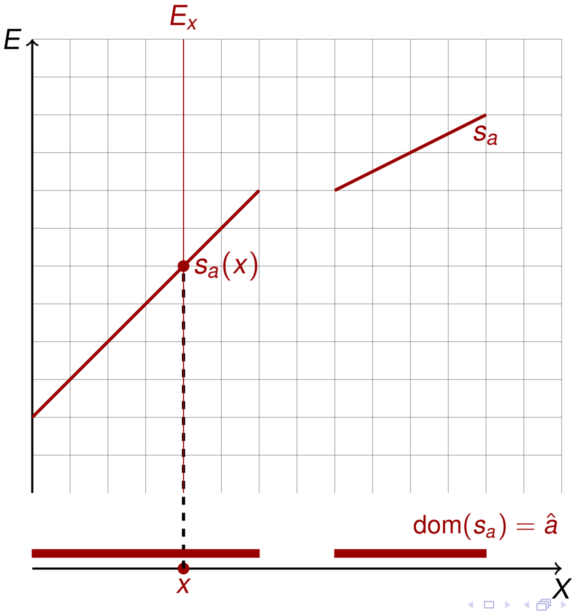

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- This map is bijective (by an induction on 'term complexity'). □

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- A morphism $\lambda : E \to E'$ is determined by the components λ_U, where U ranges over a basis of opens.
- We denote by $\mathbf{Sh}(\ell\mathbf{Priest})$ the category of sheaves over local Priestley spaces.

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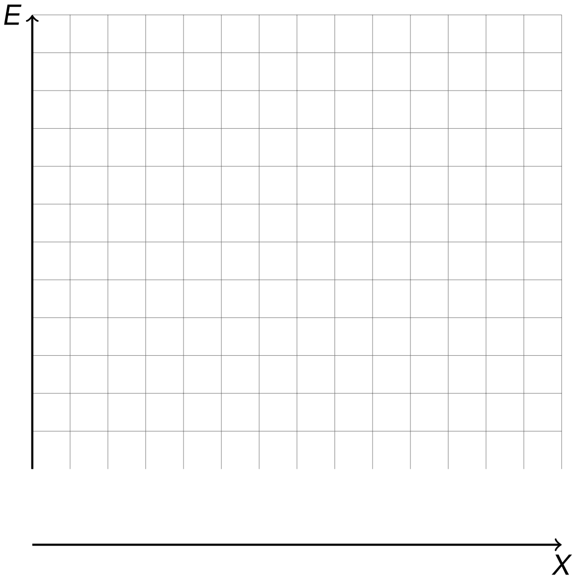
Proof.

The crucial part is the above proposition, that S is isomorphic to a skew lattice of partial sections of $S_\star$. The morphism part is a reasonably straight-forward check. □

The proof, pictorially



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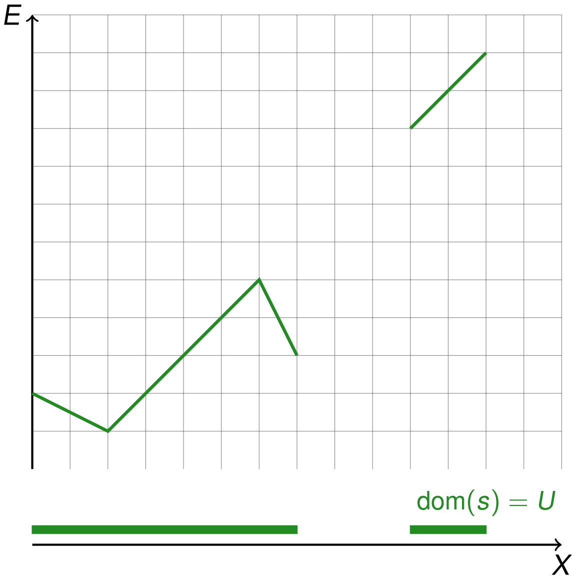

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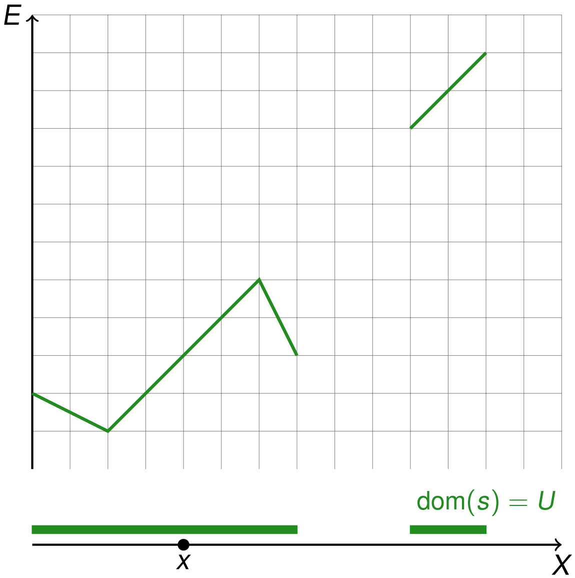

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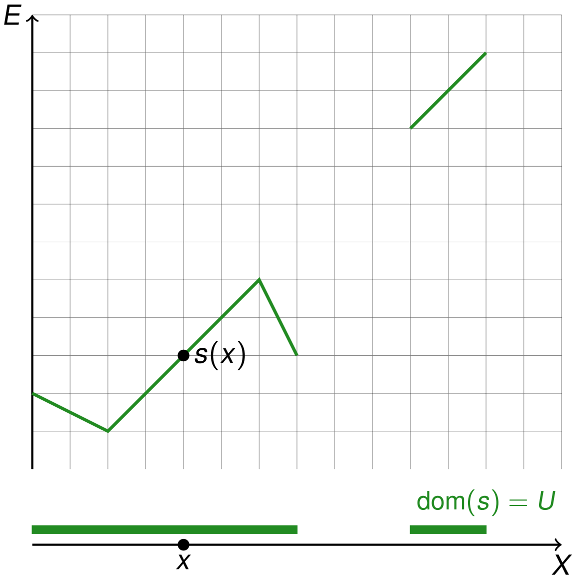

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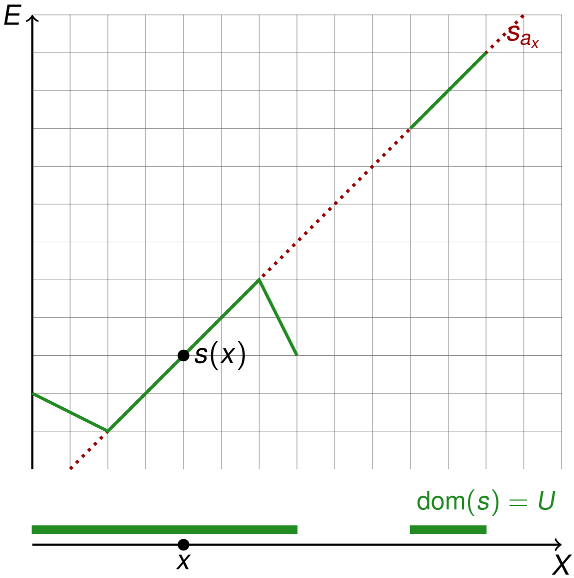

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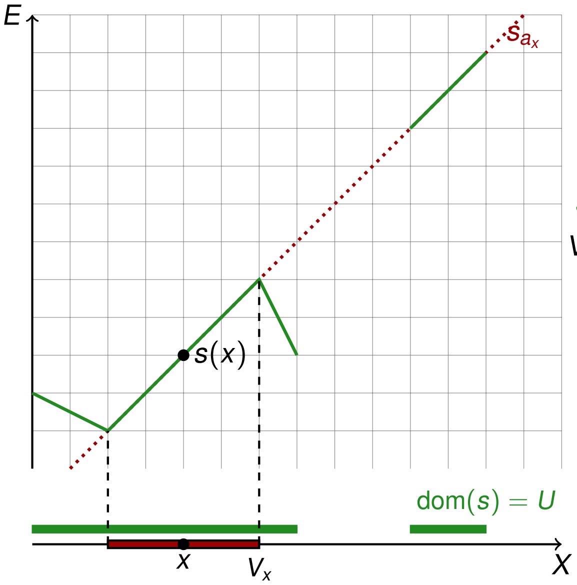

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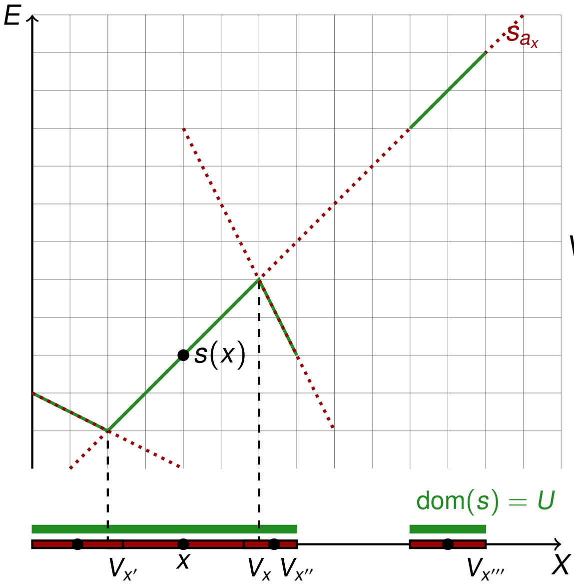

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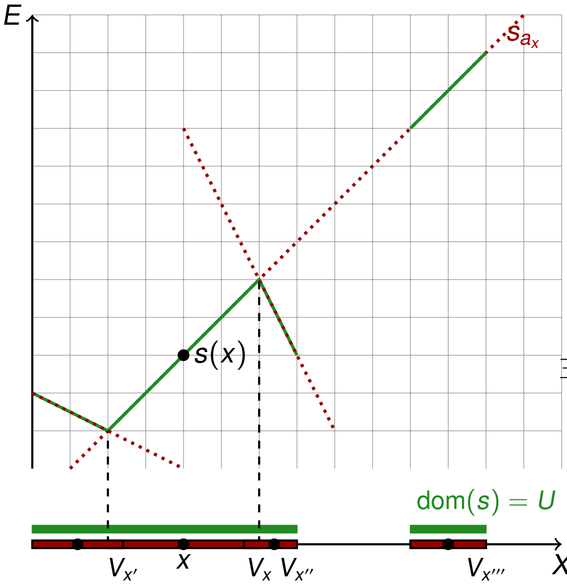

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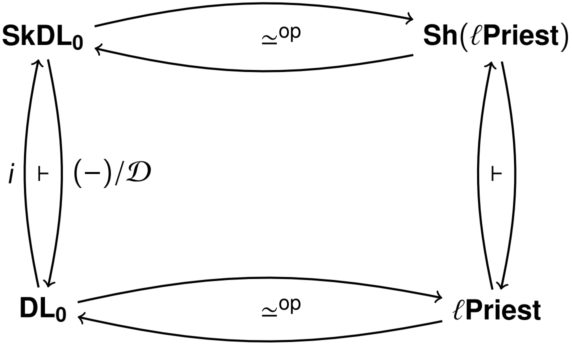
(induction) $\Rightarrow$

$\exists! a \in S : s = s_a$

Overview in a square



A non-commutative Priestley duality

Andrej Bauer*, Karin Cvetko-Vah*, Mai Gehrke°, Sam van Gool•, Ganna Kudryavtseva*

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