Discrete duality for downset lattices and their residuated operations

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Motivation

- **Labelled transition systems:**

  States $(S, \leq)$, Actions A^*, Transitions $R \subseteq S \times A^* \times S$.

- Logical description:
  1. Without negation,
  2. With converse,
  3. With $\bullet$ as primitive operation.

Presentation outline

Discrete Duality

Kripke semantics for modal logic

- **Discrete duality** for complete atomic Boolean algebras

$$\mathbf{CABA} \leftrightarrows \mathbf{Set}$$
$$\mathbb{B} \mapsto \mathrm{At}(\mathbb{B})$$
$$\mathbb{P}(X) \leftarrow\!\shortmid X$$

- Complete operator $\diamond$ on $\mathbb{B}$ $\leftrightarrow$ relation R on $At(\mathbb{B})$:

$$xR_{\diamond}y \iff x \leq \diamond y$$

- **Canonical extension:** every BA with operator embeds in a CABA with complete operator

"Canonical Extension + Discrete Duality = Frame Completeness"

Discrete Duality

Distributive lattices

- **Discrete duality** for doubly algebraic distributive lattices (Birkhoff 1940)

$$\textbf{DADLat} \leftrightarrows \textbf{Poset}$$
$$\mathbb{D} \mapsto (J(\mathbb{D}), \leq_{\mathbb{D}})$$
$$\mathcal{D}(P) \leftarrow\!\shortmid (P, \leq)$$

- Complete operator $\Diamond$ on $\mathbb{D}$ $\leftrightarrow$ relation R on $J(\mathbb{D})$ with $\leq \circ R \circ \leq = R$ (**order-compatible**)
- **Canonical extension:** every DLat with operator embeds in a DADLat with complete operator

Positive Modal Logic

Condensed Chronology

- **Dunn** (1995): modal logic in the absence of negation
- **Celani, Jansana** (1997): simple Kripke-style semantics
- **Gehrke, Nagahashi, Venema** (2004): general framework for distributive-lattice-based modal logic

Positive Modal Logic

Algebraic definition

- A **positive modal algebra** (PMA) is a distributive lattice $\mathbb{D}$ equipped with two unary operations $\Diamond$ and $\Box$, such that

$$\Box\top = \top, \qquad \Diamond\bot = \bot,$$
$$\Box(a \wedge b) = \Box a \wedge \Box b, \qquad \Diamond(a \vee b) = \Diamond a \vee \Diamond b,$$

$$\Box(a \vee b) \leq \Diamond a \vee \Box b, \qquad \Diamond a \wedge \Box b \leq \Diamond(a \wedge b).$$

- Two **interaction axioms** replacing $\Box = \neg\Diamond\neg$
- **Positive modal logic** $\mathcal{K}_+$: logic of positive modal algebras.

Positive Modal Logic

Kripke semantics for $\mathcal{K}_+$

- Call a PMA $(\mathbb{D}, \Diamond, \Box)$ **perfect** if $\mathbb{D}$ is doubly algebraic, $\Diamond$ preserves all joins, and $\Box$ preserves all meets.
- **Canonical extension magic**: every PMA embeds in a perfect PMA.
- **Consequence:** automatic Kripke-style semantics for $\mathcal{K}_+$ via discrete duality

A **Kripke frame for** $\mathcal{K}_+$ is a poset $(P, \leq)$ equipped with two binary relations $R_\Diamond$ and $R_\Box$, satisfying

$$\leq \circ R_\Diamond \circ \leq \; = R_\Diamond, \qquad \geq \circ R_\Box \circ \geq \; = R_\Box,$$

$$R_\Box \subseteq (R_\Diamond \cap R_\Box) \circ \geq, \quad R_\Diamond \subseteq (R_\Diamond \cap R_\Box) \circ \leq.$$

Residuation

Unary operation

- If $\Diamond : \mathbb{D} \to \mathbb{D}$ preserves all joins, let $\blacksquare$ its upper adjoint.
- Write $(P, \leq, R_\Diamond)$ for the corresponding Kripke frame.
- Let $\varphi \in \mathbb{D}$.

$$
\begin{aligned}
x \models \Diamond\varphi &\iff x \leq \Diamond\varphi, \\
&\iff \exists y : x\, R_\Diamond\, y \text{ and } y \models \varphi. \\
x \models \blacksquare\varphi &\iff x \leq \blacksquare\varphi, \\
&\iff \Diamond x \leq \varphi, \\
&\iff \forall y : y\, R_\Diamond\, x \text{ implies } y \models \varphi.
\end{aligned}
$$

- **Conclusion:** $R_\blacksquare = R_\Diamond^{-1}$.
- Note that $\leq \circ R_\Diamond \circ \leq = R_\Diamond$ iff $\geq \circ R_\blacksquare \circ \geq = R_\blacksquare$.

Residuation

Unary vs. Binary operations

Operations on $\mathbb{D}$	Relations on $\mathbb{P}$
Unary $\Diamond$ Upper adjoint $\blacksquare$	Binary $R_\Diamond$ with $\leq \circ R_\Diamond \circ \leq \; = R_\Diamond$ $R_\blacksquare = R_\Diamond^{-1}$
Binary $\bullet$ Upper adjoints $/$, $\backslash$	Ternary $R_\bullet$ with $\leq \circ R_\bullet \circ (\leq \times \leq) \; = R_\bullet$ ...
n-ary f Upper adjoints $f_i^\#$	$n+1$-ary R_f with $\leq \circ R_f \circ (\leq)^n \; = R_f$ ...

Residuation

Binary operation

- $\bullet : \mathbb{D}_0 \times \mathbb{D}_1 \to \mathbb{D}_2$ preserving all joins in each coordinate.
- Fixing a coordinate, $\bullet$ has upper adjoints $\backslash$ and $/$:

$$\begin{aligned} a_0 \bullet a_1 \leq a_2 &\iff a_0 \leq a_2 \backslash a_1 \\ &\iff a_1 \leq a_0 / a_2 \end{aligned}$$

- With a calculation similar to unary case:

$$\begin{aligned} &x_0 \models \varphi_2 \backslash \varphi_1 \iff \\ &\forall x_1 \forall x_2 (x_1 \models \varphi_1 \text{ and } R_\bullet(x_0, x_1, x_2) \text{ implies } x_2 \models \varphi_2) \end{aligned}$$

- **Conclusion:** $R_\backslash$ and $R_/$ are permutations of R_f
- $R_\bullet$ is order-compatible iff $R_\backslash$ is order-compatible iff $R_/$ is order-compatible

Research Question

- **Positive modal logic:**
  2 negation-dual operations ($\Diamond$, $\Box$) from 1 binary relation R
- **Residuation:**
  n residuated operations from 1 n-ary relation R
- **Question:**
  - Axiomatisation of n-ary positive modal logic with residuated operations?
  - Adding epistemic modalities?

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