

Machines, Models, Monoids, and Modal logic

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Outline

- ① Part I: Formal Languages, Automata, and Algebra
- ② Part II: Duality and Varieties of Monoids
- ③ Part III: Profiniteness, Pointlikes, and the Future

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- ① Part I: Formal Languages, Automata, and Algebra
- ② Part II: Duality and Varieties of Monoids
- ③ Part III: Profiniteness, Pointlikes, and the Future

Outline Part I

- 1 What is a formal language?
 - Alphabets and words
 - Formal languages
- 2 How to describe a formal language?
 - Automata
 - Logic
 - (Open) Problems
- 3 How to understand formal languages?
 - Boolean algebras with operators
 - Model theory
 - Monoids

Formal language theory

- A mathematical setting for analyzing computational problems.

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- Or: ... formal grammars.
- All definitions are elementary.
- Many problems are difficult, interesting, and often open.

1 What is a formal language?

- Alphabets and words
- Formal languages

Alphabets and words

- An *alphabet* is a finite set of symbols, Σ .
- A finite Σ -word is a finite sequence of elements of Σ .

Alphabets and words

Examples

- If $\Sigma = \{b, l, i, s, t, B, L, S, T\}$

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The last one can be briefly denoted as: `push_cola5`.

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- The empty word, ϵ , is a word in any alphabet.

Formal Languages

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- A (formal) Σ -*language* is a subset L of Σ^* .

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Examples

- The *empty language*, \emptyset .
- The language containing only the empty word, $\{\epsilon\}$.
- The set of all Σ -words, Σ^* .
- The set of non-empty words is a language, $\Sigma^+ = \Sigma^* \setminus \{\epsilon\}$.

Formal Languages

Examples

- Let Σ be the set of all lower-case letters, capital letters, numbers, and the symbols $!, @, \#, \$, *, (,),$ and $\%$. An example of a Σ -language is

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- Let $\Sigma = \{0, 1\}$. Three examples of $\{0, 1\}$ -languages are:
$$\text{FACTOR01} = \{w \in \{0, 1\}^* \mid w \text{ contains '01' as a factor}\}.$$
$$\text{EVENONES} = \{w \in \{0, 1\}^* \mid \text{the number of 1's in } w \text{ is even}\}.$$
$$\text{NON1} = \{0^n 1^n \mid n \geq 0\}.$$

2 How to describe a formal language?

- Automata
- Logic
- (Open) Problems

Describing formal languages

- Formal grammars
- Machines
- Logic

In this tutorial, we will focus on the last two, and we will mostly restrict to **regular** languages.

Automata

Examples

An automaton for the language BUY.



Automata

Examples

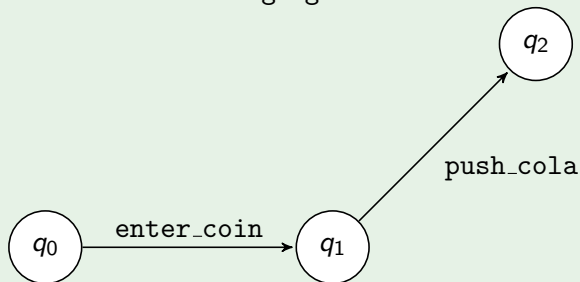
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Automata

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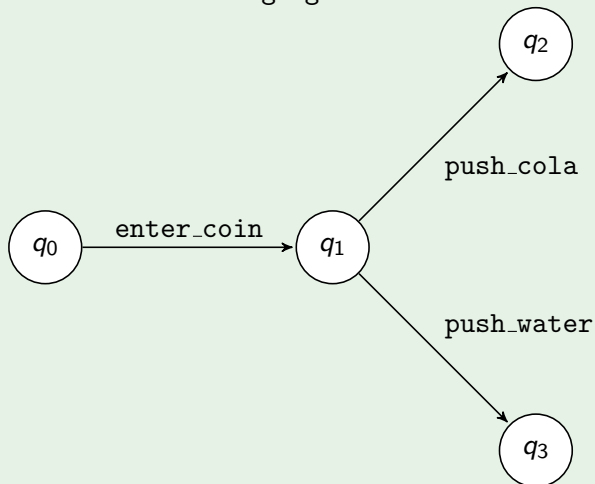
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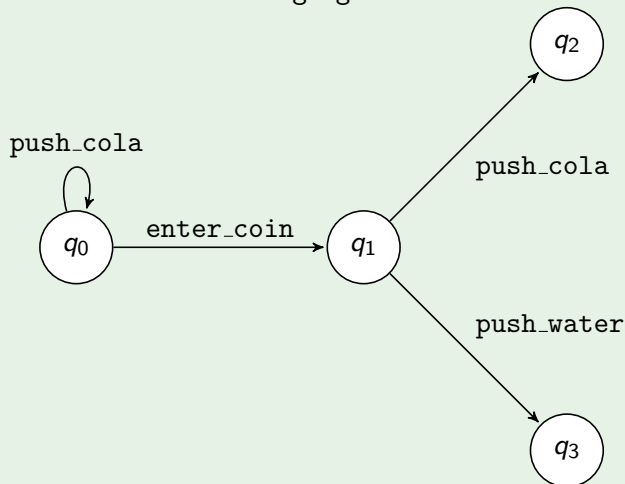
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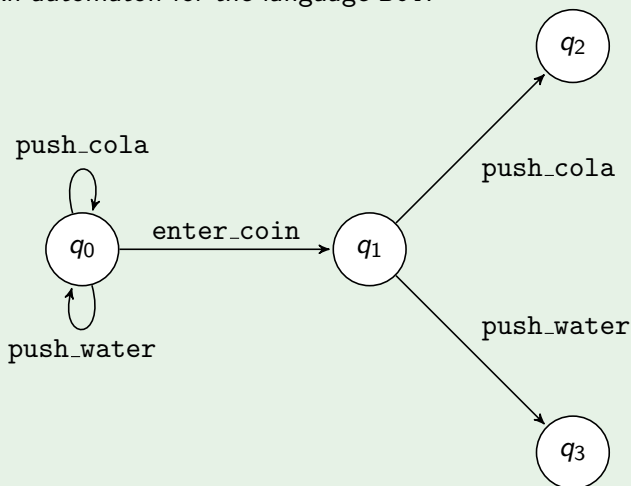
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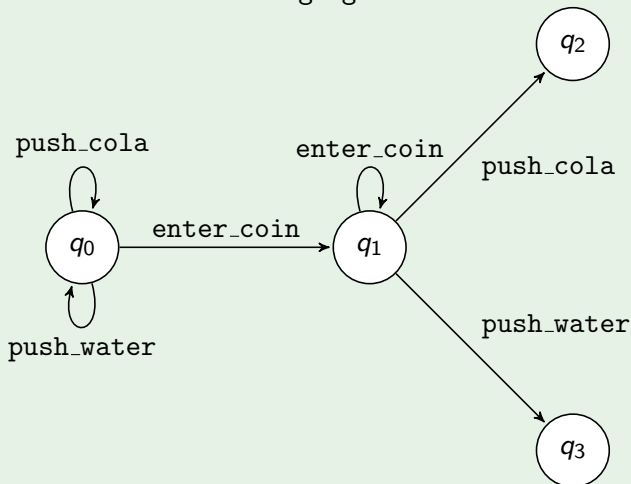
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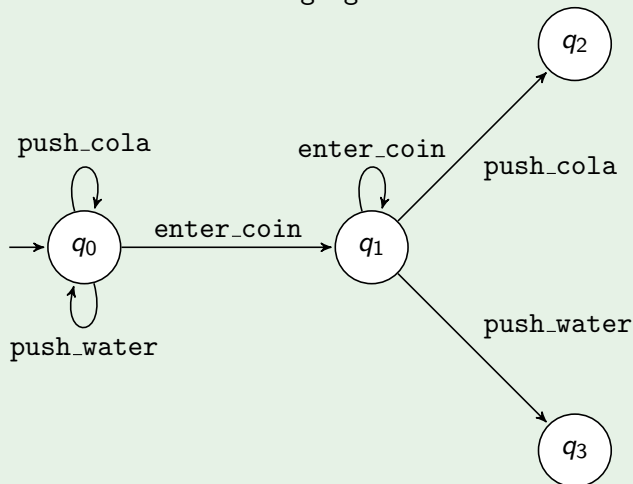
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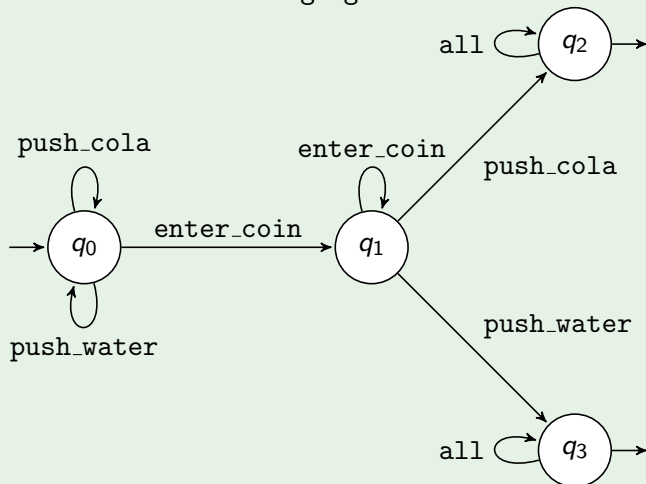
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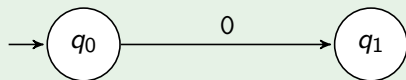


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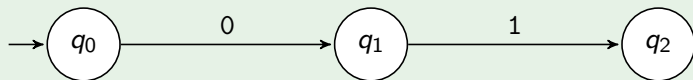


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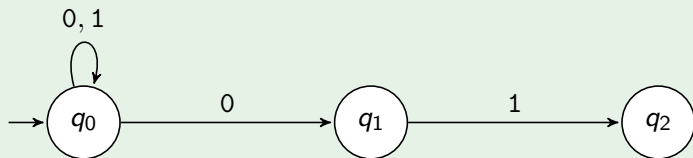


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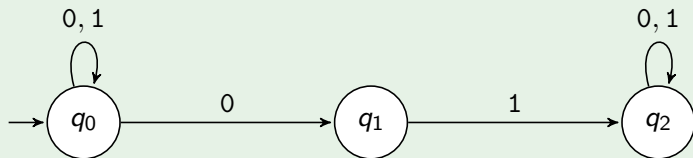


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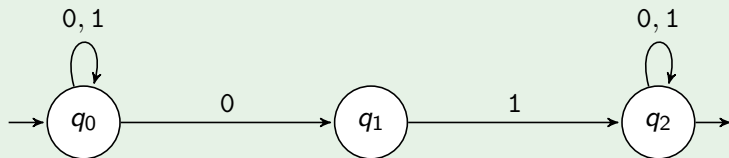


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Fact

There exists a non-deterministic finite automaton that recognizes L if, and only if, there exists a deterministic finite automaton that recognizes L .

Exercises

- 1 Describe an automaton that recognizes EVENONES.
- 2 Describe an automaton that recognizes PW.
- 3 Describe a *deterministic* automaton that recognizes FACTOR01.
- 4 (*) Is it possible to find an automaton that recognizes NON1? If no, explain why not.

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$$\text{empty} \vee \exists P [P(\text{first}) \wedge \neg P(\text{last}) \wedge \forall x \neq \text{last} (P(x) \leftrightarrow \neg P(S(x)))].$$

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- ▶ Basic propositional connectives: \wedge, \neg .
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- First Order (FO) Logic is obtained by disallowing second order variables and second order quantifiers.

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From here on, regular = MSO-definable.

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3 How to understand formal languages?

- Boolean algebras with operators
- Model theory
- Monoids

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Fact

- If L_1, L_2 are regular, then $L_1 \cup L_2$ is regular.
- If L is regular, then L^c is regular.
- If L is regular, then $a^{-1}L$ and La^{-1} are regular.

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Proof of last item.

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Corollary

If L is regular, then the set $\{w^{-1}L, Lw^{-1} \mid w \in \Sigma^\}$ is finite.*

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Proposition

A language $L \in \mathcal{P}(\Sigma^*)$ is regular if, and only if, $B(L)$ is *finite*.

Exercises

- 1 Describe the closed subalgebra generated by the $\{0, 1\}$ -language EVENLENGTH.
- 2 Let $S \subseteq \mathbb{N}$. Describe the closed subalgebra generated by the $\{1\}$ -language LENGTH $_S$ of $\{1\}$ -words of length S .
- 3 (*) When is the algebra in (2) finite?

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Semigroups and monoids

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- A *monoid* is a semigroup that contains an *identity element*, 1 , i.e., $1 \cdot x = x \cdot 1$ for all x in S .

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- For any set P , the set of functions from P to itself, $(P \rightarrow P)$, with multiplication $f \cdot g := f \circ g$.
- In particular, an NFA $\mathcal{A} = (Q, \Sigma, \delta)$ gives, for every $a \in \Sigma$, a function \diamond_a in $(\mathcal{P}(Q) \rightarrow \mathcal{P}(Q))$, defined by

$$\diamond_a(R) := \{q \mid q \xrightarrow{a} q' \text{ for some } q' \in R\}.$$

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M. Gehrke, D. Petrisan, and L. Reggio. “Quantifiers on languages and codensity monads”. In: *TACL 2017*. <https://arxiv.org/abs/1702.08841>. 2017