Proaperiodic monoids via prime models

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In earlier work [2, 3] we proved that free proaperiodic monoids can be understood as topological monoids of elementary equivalence classes of *pseudofinite words*, i.e., models of the first order theory of finite words. In particular, we showed there that every such class contains an ω-saturated member, and that algebraic operations such as concatenation, ω-power, and in fact any substitutions, are well-defined on the ω-saturated models.

Subsequently to our conference publication [2], the paper [1] gave an alternative approach to free proaperiodic monoids[1], by associating a labeled linear order of 'step points' to any element. The aim of this short note is to give a model-theoretic interpretation of the labeled linear order of [1]: it is isomorphic to the prime model for the element (up to a one-point difference).

We outline an alternative proof that such a prime model exists, independently of the results of [1]. From this, the model-theoretic fact that prime models are unique up to isomorphism immediately implies a main theorem of [1] (Thm. 8.7).

This note is merely meant as a brief announcement of these results. An article version with full proofs will be made available in due course. Only for the purposes of this note, we assume the notations of [3, 1], and we assume the model-theoretic definitions and notation of [4].

Theorem 1. *Let T be a complete theory extending the theory of pseudo-finite words. Then T has a prime model.*

Proof (sketch). By model theory (see e.g., [4, Thm. 4.2.10]), it suffices to prove that, for every n, the set of isolated n-types for T is dense in the set of all n-types for T. To this end, assume a formula $\varphi(\overline{x})$ is consistent with T. In an ω-saturated model W for T, there is a tuple $\overline{a}$ such that $W, \overline{a} \models \varphi(\overline{x}) \wedge \forall \overline{y}(\overline{y} <_{\text{lex}} \overline{x} \to \neg\varphi(\overline{y}))$, i.e., $\overline{a}$ is the lexicographically minimal witness of $\varphi(\overline{x})$. Here, the relation $<_{\text{lex}}$ is the lexicographic order on tuples, which can be defined from $<$. The n-type of the tuple $\overline{a}$ is isolated by the formula just given. $\square$

Consider an element w of $\widehat{\mathsf{F}}_{\text{AP}}(A)$, the free pro-aperiodic monoid over A. The *category of transitions* $\mathcal{T}(w)$ of w has as its objects pairs $(u, v) \in \widehat{\mathsf{F}}_{\text{AP}}(A)^2$, and morphisms $t\colon (u, v) \to (u', v')$ are elements $t \in \widehat{\mathsf{F}}_{\text{AP}}(A)$ such that $ut = u'$ and $tv' = v$. The preorder $(u, v) \preceq (u', v')$ is defined by saying there exists a morphism $(u, v) \to (u', v')$ in $\mathcal{T}(w)$, and the structure $\mathcal{L}(w)$ defined in [1, Sec. 4] is the quotient of the objects of $\mathcal{T}(w)$ by the induced equivalence relation $\equiv$ defined as $\preceq \cap \succeq$.

[1] Indeed, [1] treats a bigger class of proaperiodic monoids, but for simplicity we restrict attention to the free finitely generated ones here.

Recall that the *step points* of $\mathcal{L}(w)$ are by definition the points that are either the minimum, maximum, or have a predecessor or successor in the order. It follows from the proof of [1, Prop. 7.5] that $[(u,v)] \in \mathcal{L}(w)$ is a step point if, and only if, the endomorphism monoid of (u,v) in $\mathcal{T}(w)$ is trivial. One may show that the latter happens if, and only if, the type of (u,v) is isolated. We then obtain the following theorem. Let us denote by $\mathcal{L}'(w)$ the total order $\mathcal{L}(w)$ minus its maximum point, $(w,1)$.

Theorem 2. *Let $w \in \widehat{\mathsf{F}}_{\mathrm{AP}}(A)$ and let T be the corresponding complete theory extending the theory of pseudo-finite words. The prime model of T is isomorphic to the step points of $\mathcal{L}'(w)$.*

A key result of [1], Theorem 8.7, is that the cluster words $\mathcal{L}_c(u)$ and $\mathcal{L}_c(v)$ are isomorphic if and only if $u = v$. Note that the proof of Theorem 8.7 in [1] relies on the intricate analysis in Sections 9–11. But this is now an easy consequence of Theorem 2. Indeed, if $\mathcal{L}_c(u)$ and $\mathcal{L}_c(v)$ are isomorphic, this means in particular by Theorem 2 that the prime models for the theories of u and v are isomorphic, but then the theories are in fact the same, so $u = v$.

References

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