Frames and profinite structures

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Formalization of cohomology theories

BIRS, Banff, 22-26 May 2023

Overview

Topological spaces and frames

Coherence, or: how to make it profinite

Ordered spaces

Adding (co)algebraic structure

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Duality between points and opens

A point x of a topological space X determines a collection of open neighborhoods

$$\epsilon(x) := \{U \in \mathcal{O}(X) \mid x \in U\} .$$

The function ϵ maps X to its 'double dual'.

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The function ϵ maps X to its 'double dual'.

But what is the 'dual' of a topological space?

Frames

A frame is a complete lattice $(L, \leq, \bigvee, \wedge, 1)$ such that

$$u \wedge \left(\bigvee S\right) = \bigvee_{v \in S} (u \wedge v)$$

for any $u \in L$ and $S \subseteq L$.

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An element $u \in L$ is compact if for any $S \subseteq L$, $u \leq \bigvee S$ implies $u \leq \bigvee F$ for some finite $F \subseteq S$. L is compact if 1 is compact.

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A map $f\colon X \to Y$ gives a homomorphism $f^{-1}\colon \mathcal{O}(Y) \to \mathcal{O}(X)$.
A homomorphism between frames is a $\wedge, 1, \bigvee$ preserving function.

Examples of frames

- The open sets $\mathcal{O}(X)$ of any topological space X.

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$$\bigvee_{i \in I} J_i = \sqrt{\bigoplus_{i \in I} J_i}$$

(Also appears in local cohomology, see for example Mathlib PR #19061)

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- The regular open subsets of a compact Hausdorff space.

$$\bigvee_{i \in I} R_i = \overline{\bigcup_{i \in I} R_i}^{\circ}$$

The dual space of a frame

A homomorphism

$$x\colon L \to \mathbf{2}$$

to the two-element frame $\mathbf{2} = \mathcal{O}(*) = \{0, 1\}$ is called a point of L.

(Some people look at $\mathbf{Frm}^{\mathrm{op}}$ rather than **Frm**, then call the objects *locales*, and denote **2** by **1**.)

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The set of points of L, $\operatorname{pt} L$, carries a topology

$$\{\widehat{u} \;:\; u \in L\}$$

where

$$\widehat{u} := \{x \in \operatorname{pt} L \;\mid\; x(u) = 1\} \;.$$

A dual adjunction

We have an adjunction

$$\mathrm{pt}\colon \mathbf{Frm}^{\mathrm{op}} \leftrightarrows \mathbf{Top}\colon \mathcal{O}$$

with unit and co-unit

$$\epsilon_X\colon X \to \mathrm{pt}\,\mathcal{O}X \qquad \text{and} \qquad \eta_L\colon L \to \mathcal{O}\,\mathrm{pt}\,L\ .$$

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The fixed points on the left are the spatial frames and on the right the sober spaces.
Sober: T_0 and every irreducible closed set has a generic point.
Fact (in Mathlib): Hausdorff $\Rightarrow$ sober.

Examples of dual spaces

- Points of $\mathcal{O}X$ correspond to irreducible closed sets of X:

$$x\colon \mathcal{O}X \to \mathbf{2} \quad \longleftrightarrow \quad X \setminus \left(\bigcup\{U \mid x(U) = 0\}\right).$$

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- Points of $\mathcal{R}\mathcal{O}X$ are … there may not be any.

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A profinite set is any topological space that is a cofiltered limit of objects DF with F a finite set.

Proposition

A topological space X is a profinite set if, and only if, X is compact and totally separated, that is, for any $x, y \in X$,

$$\text{if } x \neq y \text{ then there is a clopen } K \subseteq X \text{ such that } x \in K \text{ and } y \notin K.$$

The category of Profinite Types

We construct the category of profinite topological spaces, often called profinite sets -- perhaps they could be called profinite types in Lean.

The type of profinite topological spaces is called `Profinite`. It has a category instance and is a fully faithful subcategory of `TopCat`. The fully faithful functor is called `Profinite.toTop`.

Implementation notes

A profinite type is defined to be a topological space which is compact, Hausdorff and totally disconnected.

TODO

0. Link to category of projective limits of finite discrete sets.
1. finite coproducts
2. Clausen/Scholze topology on the category `Profinite`.

Tags

profinite

```
structure Profinite
      :
  Type (u_1+1)
```

ink source

The underlying compact Hausdorff space of a profinite space.

```
toCompHaus : CompHaus
```

A profinite space is totally disconnected.

```
IsTotallyDisconnected : TotallyDisconnectedSpace ↑toCompHaus.toTop
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The type of profinite topological spaces

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A First Mile-Stone™

Formalize a proof that the following categories are equivalent:

1. Compact totally separated topological spaces
2. Cofiltered limits in **Top** of objects DF with F finite
3. Finite-limit-preserving functors **FinSet** $\to$ **Set**
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Roadmap.

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3 $\iff$ 4 is 'just category theory' (famous last words).

Stone duality for Boolean algebras

Theorem (Stone 1937)

$$\mathbf{BA}^{\mathrm{op}} \simeq \mathbf{Pro\,FinSet}\ .$$

Proof. Given the First Mile-Stone™, this is easy:

- $\mathbf{FinBA}^{\mathrm{op}} \simeq \mathbf{FinSet}$,
- $\mathbf{Ind}(\mathbf{FinBA}) \simeq \mathbf{BA}$,
- $\mathbf{Ind}(\mathbf{C})^{\mathrm{op}} \simeq \mathbf{Pro}(\mathbf{C}^{\mathrm{op}})$.

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(Not Stone's original proof. No ultrafilters, at least not explicitly.)

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- $\mathbf{Ind}(\mathbf{C})^{\mathrm{op}} \simeq \mathbf{Pro}(\mathbf{C}^{\mathrm{op}})$.

What is $\mathbf{Pro}\,\mathbf{FinT_0}$?

Spectrality and Coherence

Proposition

A topological space X is a projective limit of finite T_0 spaces if, and only if, it is spectral, that is, compact, sober, and has a basis of compact-open sets which is closed under finite intersections.

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A space X is spectral if, and only if, the frame $\mathcal{O}(X)$ is coherent, that is, its compact elements are a $\bigvee$-dense sublattice.

Examples of spectral spaces

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Proof. Interesting. □

Proposition (A more feasible sub-goal)

Every finite distributive lattice is the lattice of finitely generated radical ideals of some ring R.

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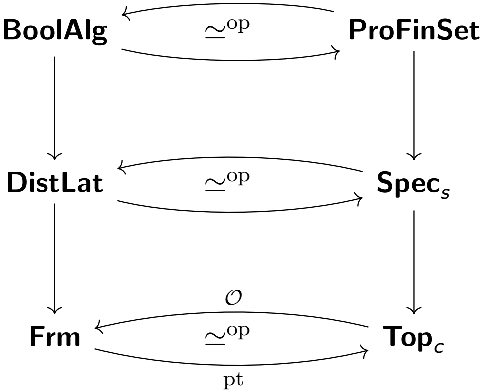
However: not every continuous function $X \to Y$ between spectral spaces factors through the limit diagram!

A function $f: X \to Y$ between spectral spaces is called spectral if $f^{-1}(K)$ is compact-open for any compact-open set $K \subseteq Y$.

Taking stock: Stone's dualities



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Any spectral topology σ on a set X has an inverse topology σ^∂, which is also spectral, and has the inverse specialization order. The patch topology σ^p is the join of σ and σ^∂.

Proposition

The partially ordered topological space $(X, \sigma^p, \rightsquigarrow)$ is compact and totally order-separated: for any $x, y \in X$, if $x \not\lesssim y$, then there is a clopen $\rightsquigarrow$-up-set $K \subseteq X$ such that $x \in K$ and $y \notin K$.

Such a structure is called a Priestley space.

Spectral and Priestley

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Proposition

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The Hausdorff spectral spaces ($=$ profinite sets) correspond to the Priestley spaces with trivial specialization order.

Profinite posets

As with profinite sets, there is a fully faithful functor

$$D\colon \textbf{FinPoset} \to \textbf{Priestley}$$

which maps a finite poset $(P, \leq)$ to $(P, \tau_{\mathrm{discrete}}, \leq)$.

Proposition

The category of Priestley spaces is equivalent to the Pro-completion of **FinPoset**.

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Proposition

The profinite set underlying $\widehat{\mathbb{Z}}$ is $\operatorname{spec} A$, where $A \leq \mathbf{2}^{\mathbb{Z}}$ is the Boolean algebra generated by arithmetic progressions. The group structure of $\widehat{\mathbb{Z}}$ is dual to the shift map on A.

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Useful for proving the Skolem theorem: the zero set of a linear recurrence (in $\mathbb{Z}$) is a finite union of arithmetic progressions, up to a finite error. (A nice formalization project?)

A connection to recognizable sets

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Free profinite monoids naturally appear here:

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The free profinite monoid on $\{0,1\}$ is the spectrum of the Boolean algebra of regular languages (extended with coalgebraic structure).

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More generally:

Theorem (Gehrke)

The profinite completion of an algebraic structure A is the extended spectrum of the Boolean algebra of recognizable sets in A.

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Definition

A profinite λ-term is a point of the algebra of λ-recognizable sets.

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(Mathematical) WIP: extend the usual profinite monoid methods to this setting.

Summary

- Stone duality: mostly linking up some existing parts of the library, no big roadblocks expected.
- Profinite posets: some more work but doable.
- Potential new application domains (in addition to Condensed Math): Hochster, Skolem.
- Adding (co)algebraic structure: a longer-term project.

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