

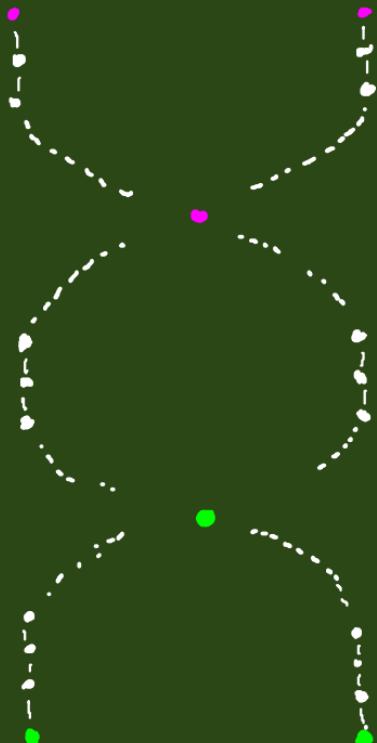
# Priestley duality for MV algebras and beyond

BLAST

June 13, 2021

Stone-Priestley session

Sam van Gool, joint work with Wesley Fussner, Mai Gehrske and Vincenzo Marra



Priestley dual space of the algebra  
of  $\mathbb{Z}$ -PL functions  $[0,1] \rightarrow [0,1]$   
localized at  $\frac{1}{2}$

Forum Math. (2021),  
arXiv: 2002.12715v3

## Plan

1. Priestley duality extended to double quasi-operators
2. Applying correspondence theory
3. The case of MV-algebras

## Plan

1. Priestley duality extended to double quasi-operators

→ How to dualize operations of implication type  
that respect both  $\wedge$  and  $\vee$ ?

2. Applying correspondence theory

3. The case of MV-algebras

## Plan

1. Priestley duality extended to double quasi-operators

→ How to dualize operations of implication type  
that respect both  $\wedge$  and  $\vee$ ?

2. Applying correspondence theory

→ How to dualize equational axioms?

3. The case of MV-algebras

## Plan

1. Priestley duality extended to double quasi-operators

→ How to dualize operations of implication type  
that respect both  $\wedge$  and  $\vee$ ?

2. Applying correspondence theory

→ How to dualize equational axioms?

3. The case of MV-algebras

→ What does this yield in the special case of MV?

## 1. Priestley duality extended to double quasi-operators

Def. Let  $L$  a bounded distributive lattice.

A double quasi-operator of reverse implication type,

or, a "minus", is a binary function  $\ominus : L \times L^o \rightarrow L$

satisfying the equations:

$$\begin{aligned} -\ominus b \\ \text{is a hom.} \\ \text{except for } 1 \end{aligned} \left\{ \begin{array}{l} (a_1 \vee a_2) \ominus b = (a_1 \ominus b) \vee (a_2 \ominus b) \\ (a_1 \wedge a_2) \ominus b = (a_1 \ominus b) \wedge (a_2 \ominus b) \\ 0 \ominus b = 0 \end{array} \right.$$

$$\left. \begin{array}{l} a \ominus (b_1 \vee b_2) = (a \ominus b_1) \wedge (a \ominus b_2) \\ a \ominus (b_1 \wedge b_2) = (a \ominus b_1) \vee (a \ominus b_2) \\ a \ominus 1 = 0 \end{array} \right\} \begin{array}{l} a \ominus - \\ \text{is an anti-hom.} \\ \text{except for } 0 \nmid 1 \end{array}$$

A pair  $(L, \ominus)$  is a  $\ominus$ -algebra.

(Throughout the talk, we moreover assume  $a \ominus 0 = a$ ,  
but this is inessential for the theory to work.)

Theorem (Gehrke-Jónsson)

$L$  a bounded DL with Priestley space  $X$ .

Any minus operator admits two canonical liftings

$$\Theta^\sigma, \Theta^\pi : \mathcal{D}X \times (\mathcal{D}X)^{\text{op}} \rightarrow \mathcal{D}X$$

to minus operators on  $\mathcal{D}X$ ,

and any equation on  $(L, \Theta)$  lifts to both  $(\mathcal{D}X, \Theta^\sigma)$  and  $(\mathcal{D}X, \Theta^\pi)$ .

But:  $\Theta^\sigma$  respects  $\vee$ , while  $\Theta^\pi$  respects  $\wedge$  in both coordinates.

Example.  $L \models a \wedge b \leq a \ominus (a \ominus b) \approx \text{"MV6"}$   
implies

$$\mathcal{D}X \models a \wedge b \leq a \Theta^\pi (a \Theta^\pi b).$$

Recall  $X \xhookrightarrow{\mu} \mathcal{D}X$  via  $\mu_x := X \setminus \uparrow_x$ .  
 $M = \text{im}(\mu)$ .

For any  $y \in X$ ,  $-\ominus^\sigma \mu_y : \mathcal{D}X \rightarrow \mathcal{D}X$  preserves  $\vee$ , and  
 therefore has an upper adjoint  $-+ \mu_y : \mathcal{D}X \rightarrow \mathcal{D}X$  defined by

$$u \ominus^\sigma \mu_y \leq \sigma \iff u \leq \sigma + \mu_y.$$

$\diamond(-, -)$

}

Fact. For any  $x, y \in X$ ,  $\mu_x + \mu_y \in M \cup \{1\}$ .  $+ \rightarrow R(-, -, -)$

Recall  $X \xrightarrow{\nu} \mathcal{D}X$  via  $\nu_x := \downarrow x$ .

$$\mathbb{J} = \text{im}(\nu).$$

For any  $y \in X$ ,  $- \ominus^\pi \nu y : \mathcal{D}X \rightarrow \mathcal{D}X$  preserves  $\wedge^{\text{non-}\phi}$ , and

therefore has a p. lower adjoint  $- * \nu y : \mathcal{D}X \rightarrow \mathcal{D}X$  defined by

$$u \ominus^\pi \nu y \geqslant \sigma \iff (u \geqslant \sigma * \nu y \text{ and } u \leqslant 1 \ominus^\pi \nu y).$$

Fact. For any  $x, y \in X$ ,  $\nu_x * \nu y \in \mathbb{J} \cup \{0\}$ , if  $\nu x \leq 1 \ominus^\pi \nu y$ .

Thus:  $\ominus^r$  yields  $+$  on the dual,

$\ominus^{\text{II}}$  yields  $*$  on the dual.

Both  $+$  and  $*$  are partial,  
but we avoid mention  
of their domains here,  
see paper for full details.

Prop (+)  $x * y = \inf \{ x + w : w \notin y \}$

→ How does  $+$  "feel" that it is dual to a double minus?

Prop (+) For any  $x \in X$ , the translation  $x + -$  has a  
totally ordered image and an upper adjoint,  $k(x, -)$ .

Note: N. Martínez previously found the function  $x \mapsto k(x, x)$  "in the wild" in the 90s.  
(for MV-algebras)

Theorem (Extended Priestley duality for  $\Theta$ -algebras.)

The category of  $\Theta$ -algebras is dually equivalent to a category of  $\Theta$ -spaces:

$(X, i, +, *)$  where

$i$  records what  
 $a \mapsto 1\Theta a$   
does.

$X$  a Priestley space,

$i : X^{\text{op}} \rightarrow X$  cts. order-reversing function,

$+ : X^2 \rightarrow X$  upper cts. order-preserving partial fn.,

$* : X^2 \rightarrow X$  lower cts. order-preserving partial fn.,

$(*)+$  and  $(+)$  hold.

effect algebras.

$$R \subseteq X^3$$



## 2. Applying Correspondence Theory

Building on the foundation of our duality for  $\Theta$ -algebras, we may now add axioms to obtain dualities for subvarieties.

Convenient notations :  $\neg a := 1 \Theta a$

$$a \oplus b := \neg(\neg a \Theta b).$$

Example axioms : (1)  $\neg\neg a = a$

(2)  $a \oplus b = b \oplus a$

(3)  $(a \oplus b) \oplus c = a \oplus (b \oplus c).$

(4)  $\oplus$  is upper residual of  $\Theta$ .



$$(a \oplus b) \oplus b \leq a \text{ and } a \leq (a \oplus b) \oplus b.$$

- |     |   |
|-----|---|
| (1) | $\neg\neg a = a$                                  |
| (2) | $a \oplus b = b \oplus a$                         |
| (3) | $(a \oplus b) \oplus c = a \oplus (b \oplus c)$ . |
| (4) | $\oplus$ is upper residual of $\ominus$ .         |

For each of these examples, if a  $\Theta$ -algebra  $L$  satisfies the equation (n)  
then so does  $(DX, \Theta^\pi)$ .

Moreover, (4) implies that  $\oplus^\pi$  coincides with + (where defined).

Now a simple and standard application of correspondence theory  
yields duals for these axioms.

Theorem. A  $\Theta$ -algebra  $L$  satisfies

- (1)  $\neg\neg a = a$
- (2)  $a \oplus b = b \oplus a$
- (3)  $(a \oplus b) \oplus c = a \oplus (b \oplus c)$ .
- (4)  $\oplus$  is upper residual of  $\Theta$ .

if and only if

its dual  $\Theta$ -space  $(X, i, +, *)$  satisfies

- (1)  $i^2 = i$
  - (2)  $+$  commutative
  - (3)  $+$  associative
  - (4)  $i$  respects translation by  $+$ :  
 $x + y \leq z \text{ iff } i(z) + y \leq i(x).$
- } pre-MV-space

### 3. The case of MV-algebras

(or: the use of \*)

Fact. A  $\Theta$ -algebra is (term-equivalent to) an MV-algebra iff it satisfies (1)-(4) and  $a \wedge b \leq a \Theta (a \Theta b)$  (MV6).

As we remarked in the beginning, MV6 lifts to

$$\mathcal{D}X \models u \wedge v \leq u \ominus^\pi (u \ominus^\pi v)$$

or equivalently

$$\mathcal{D}X \models \neg u \wedge v \leq (u \oplus^\pi v) \ominus^\pi u.$$

$\forall u, v$

Now, standard correspondence arguments let us rewrite this into

$$\forall x, y \in X, \quad \nu x \leq \neg \nu y \Rightarrow \nu x \leq (\nu x \oplus^\pi \nu y) \ominus^\pi \nu y.$$

To isolate the blue term, we must do:

$$\forall x, y \in X, \quad \nu x \leq \neg \nu y \Rightarrow \nu x * \nu y \leq \nu x \oplus^\pi \nu y.$$

This is where  $*$  appears.

The condition can then be further rewritten to replace  $\oplus^\pi$  by  $+$ ,  
using standard methods.

Theorem. The category of MV-algebras is dually equivalent to the full subcategory of  $\Theta$ -spaces on MV-spaces, i.e., pre-MV-spaces satisfying the additional (first-order!) property:

$\forall(x,y) \in \text{dom}(*)$ ,  $x', y' \in X$ , if  $y' \neq y$  and  $x' * y' \leq x * y$   
 then  $x' \leq x$ .

"mixed cancellativity of  $*$  and  $+$ "?

## Take-aways and directions

- Extended Priestley duality with partial functions (instead of relations) on the dual is possible in the general setting of  $\Theta$ -algebras.
- Correspondence & canonicity on top of this yields concrete results for subvarieties like MV.  
(Also see my first paper with Gehrke & Marra '14)
- Other example applications : - MTL-algebras (an example in the paper)
  - residuated lattices of Scott-continuous functions on complete chains  
(cf. recent work with Guatto, Metcalfe & Santachi).

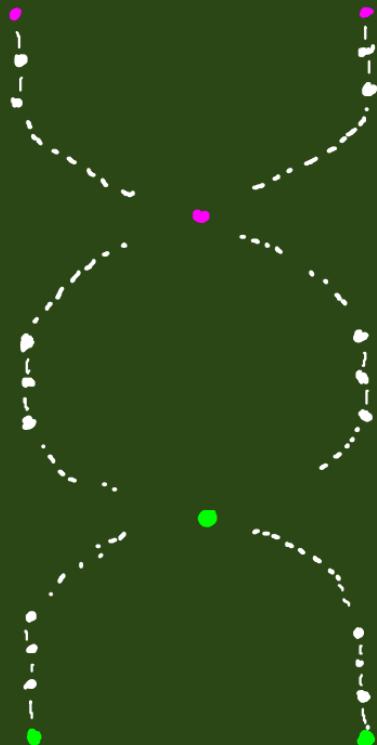
# Priestley duality for MV algebras and beyond

BLAST

June 13, 2021

Stone-Priestley session

Sam van Gool, joint work with Wesley Fussner, Mai Gehrske and Vincenzo Marra



Priestley dual space of the algebra  
of  $\mathbb{Z}$ -PL functions  $[0,1] \rightarrow [0,1]$   
localized at  $\frac{1}{2}$