What is an existentially closed Heyting algebra and what does it have to do with automata?

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IITK Logic Webinar
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About this talk

- **Objective.** Show an instance of interaction between
  logical algebra, model theory and automata.

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  logical algebra, model theory and automata.
- **Format.** Part tutorial (∼20m), part research talk (∼40m).

Overview

Part I: A tutorial on algebra in logic

Regular languages and logic

Logical algebra

Part II: Model completeness in logical algebra

Model completeness and model companions

MSO on omega is the model companion of LTL

An excursion to trees

An excursion to Heyting

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Three ways of describing a regular language

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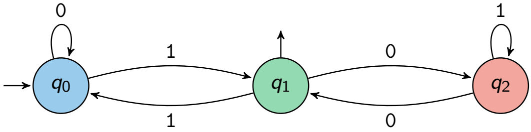

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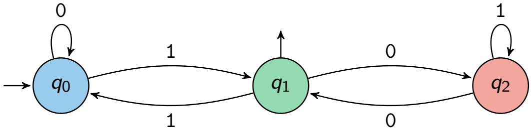

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- Solution 2: a homomorphism $\varphi\colon \{0,1\}^* \to S_3$ defined by

$$0 \mapsto (1\,2), \quad 1 \mapsto (0\,1).$$

Answer yes iff the permutation $\varphi(w)$ sends 0 to 1.

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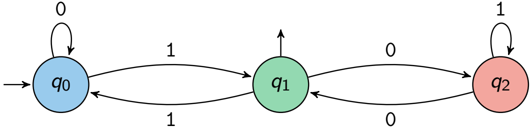

Answer yes iff A accepts w.

- Solution 3: a predicate formula φ describing A:

$\exists Q_0 \exists Q_1 \exists Q_2 (Q_0(\texttt{first}) \land Q_1(\texttt{last}) \land$

$\forall x[\texttt{0}(x) \land Q_0(x) \to Q_0(\texttt{S}x)] \land [\texttt{1}(x) \land Q_0(x) \to Q_1(\texttt{S}x)] \land \dots).$

Answer yes iff w satisfies the formula φ.

Predicate logic on finite words

- **Syntax.** Monadic Second Order (MSO) logic over $<$, Σ.
  - Basic propositional connectives: $\wedge$, $\neg$.
  - Quantification over first-order variables x, y, ... and one-place (monadic) second-order variables P, Q, ...
  - Relational signature: $x < y$, $\mathtt{a}(x)$ for $a \in \Sigma$.

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- **Semantics.** A finite word $w = a_1 \ldots a_n$ gives a structure W.
  - The underlying set of W is $\{1, \ldots, n\}$.
  - The natural linear order $<^W$ interprets the binary predicate $<$.
  - For every letter $a \in \Sigma$, $\mathrm{a}^W := \{i \in \{1, \ldots, n\} \colon a_i = a\}$.

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- For a sentence φ, $L_\varphi := \{w \in \Sigma^* \mid w \models \varphi\}$.
- Shortcuts $\mathtt{S}x$, $\mathtt{first}$, $\mathtt{last}$, $\subseteq$ are MSO-definable.

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- This phenomenon of 'algebra on two levels' is a special instance of Stone duality.

Boolean algebras of languages

- The set Sent(Σ) of all MSO-sentences over a fixed finite alphabet Σ carries a preorder, $\vdash$:

$$\varphi \vdash \psi \iff \text{for every finite word } W, \text{ if } W \models \varphi, \text{ then } W \models \psi.$$

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- The quotient of Sent(Σ) under $\vdash$-equivalence is a *Boolean algebra*, $R(\Sigma)$:

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- **Theorem (Büchi 1960).** The image of the injection

$$R(\Sigma) \hookrightarrow \mathcal{P}(\Sigma^*), \quad \varphi \mapsto \{W \in \Sigma^* \mid W \models \varphi\}$$

consists of the regular Σ-languages.

Logical algebra

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- The abstractions of logical algebra allow us not to think about concrete formulas, but also treat other entities, such as languages and other sets, as if they are formulas.

Boolean algebras as lattices

- A bounded lattice is a tuple $(L, \leq, \vee, \wedge, \bot, \top)$, where $\leq$ is a partial order, and for any $a, b \in L$, $a \vee b = \sup\{a, b\}$, $a \wedge b = \inf\{a, b\}$, $\bot = \sup \emptyset$, and $\top = \inf \emptyset$.

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- **Exercise.** Boolean algebras are term-equivalent with idempotent commutative rings with unit.

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- The familiar *truth tables* compute these values under assignment to the two-element Boolean algebra $\{\bot, \top\}$.
- In other logics, a single finite algebra is *not* enough.

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- *Intuitionistic logic* ... Heyting algebras.

Heyting algebras

- A Heyting algebra is a tuple $(H, \wedge, \vee, \bot, \top, \to)$, where
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- **Exercise (non-trivial).** There are infinitely many non-equivalent terms in a single variable x.

Summary of Part I: Tutorial

- Monadic second order logic has the same expressive power as finite automata.
- Boolean algebras are abstract algebraic models for propositional logic.
- Generalizing Boolean algebras in various directions (modal, temporal, and Heyting algebras) allows one to talk about different logics in one algebraic framework.

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Part II: Model completeness in logical algebra

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- Usually, the structures studied are classical: fields, groups, ...
- In this work, we apply model theory to structures *from logical algebra*, that is, to Boolean algebras, to Heyting algebras, to LTL algebras, and more.

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- A T-structure A is existentially closed* if any existential sentence that becomes true in some T-structure extending A already holds in A.
- This property is often first order definable:
  - *Linear orders without endpoints:* density;
  - *Boolean algebras:* atomless;
  - *Heyting algebras:* I will sketch this a few slides from now.

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Model companion

A first order theory T^* which captures the existentially closed models for a universal theory T is called a model companion of T.

Theorem.
The theory T^*, if it exists, is the unique theory such that:

1. T and T^* believe the same universal sentences;

2. For any sentence φ, there is an existential sentence φ' such that T^* believes $\varphi \leftrightarrow \varphi'$.

Robinson, 1963

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   T and T^* are co-theories

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   T^* is model complete

Robinson, 1963

Overview

MSO on omega is the model companion of LTL

Joint work with Silvio Ghilardi (Milan).

Model companions and languages

Theorem.

The first order theory T^* of an algebra for word languages, $\mathcal{P}(\omega)$,

is the model companion of

a theory T of algebras for a fragment of linear temporal logic.

"MSO on ω is the model companion of LTL"

Ghilardi & G. JSL 2017

For convenience, we switch from finite words to ω-words for this part.

The theory T^*: the generic LTL-algebra

- The Boolean algebra $\mathcal{P}(\omega)$ carries temporal operators:
  - $\mathbf{X}a := \{t \in \omega \mid t+1 \in a\}$,
  - $\mathbf{F}a := \{t \in \omega \mid \exists t' \geq t \colon t' \in a\}$,
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- The theory T^* is the theory, $\mathrm{Th}(\mathcal{P}(\omega))$, of this (single) structure, in the signature $\{\vee, \wedge, \bot, \top, \neg, \mathbf{X}, \mathbf{F}, \mathbf{I}\} \cup \{=\}$.

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- **Exercise.** Let φ be an $(\mathbf{X}, \mathbf{F}, \mathbf{I})$-formula in variables $x_1, \ldots, x_n$. For each $1 \leq i \leq n$, let $X_i \subseteq \omega$. For any $t \in \omega$, we have

$$t \in \varphi^{\mathcal{P}(\omega)}(\overline{X}) \text{ iff } \varphi \text{ holds at } t \text{ in the Kripke model } (\omega, \overline{X}).$$

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- **Theorem.** $T^* = \mathrm{Th}(\mathcal{P}(\omega))$ is the model companion of T.

The theory T: general LTL-algebras

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  - $\mathbf{B} = (B, \vee, \wedge, \neg, \bot, \top)$ is a Boolean algebra;
  - $\mathbf{X}$ is an endomorphism of B;
  - $\mathbf{I}$ is an atom, $\mathbf{XI} = \bot$, and $\mathbf{I} \leq \mathbf{X}a$ when $a \neq \bot$.
  - for any $a \in B$, $\mathbf{F}a$ is the least fixed point of $x \mapsto a \vee \mathbf{X}x$, i.e.,
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- T is the theory of the linear temporal algebras.

- **Theorem.** $T^* = \mathrm{Th}(\mathcal{P}(\omega))$ is the model companion of T.
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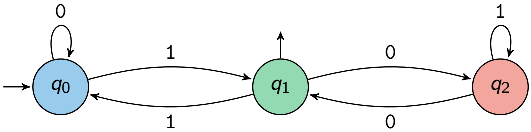
  - Co-theories: a non-trivial **exercise**.
  - Model completeness of T^*: automata!

Recall our first example

- A programming problem: given a natural number in binary, $w \in \{0,1\}^*$, determine if w is congruent 1 modulo 3.
- Solution 1: a (deterministic) automaton A:



Answer yes iff A accepts w.

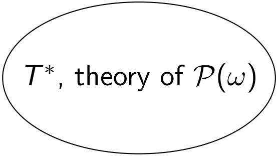
- Solution 3: a predicate formula φ describing A:

$\exists Q_0 \exists Q_1 \exists Q_2 (Q_0(\texttt{first}) \wedge Q_1(\texttt{last}) \wedge$

$\forall x [\texttt{0}(x) \wedge Q_0(x) \rightarrow Q_0(\texttt{S}x)] \wedge [\texttt{1}(x) \wedge Q_0(x) \rightarrow Q_1(\texttt{S}x)] \wedge \dots).$

Answer yes iff w satisfies the formula φ.

Proving model completeness with automata



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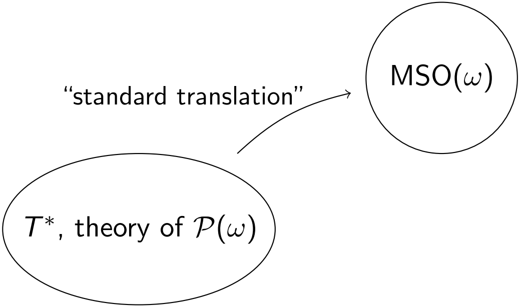

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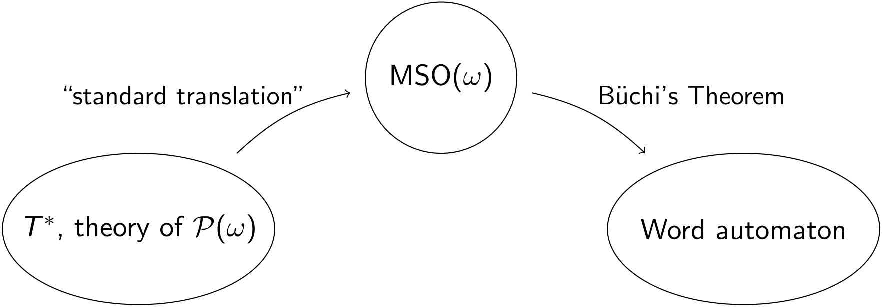

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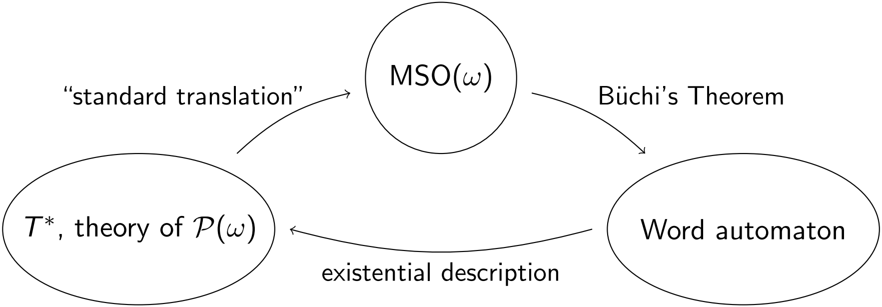

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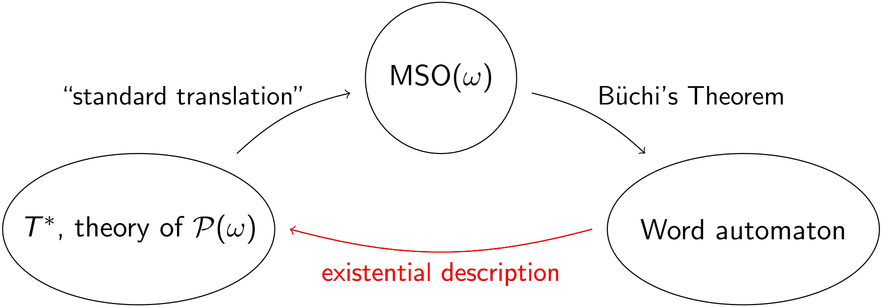

An existential description of a word automaton

- Let $A = (Q, \Sigma, \delta, q_0, F)$ be a **word automaton** over a finite alphabet Σ, i.e., a function $\delta\colon Q \times \Sigma \to \mathcal{P}(Q)$, an initial state $q_0 \in Q$ and a subset $F \subseteq Q$ of final states.

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- Write $\Sigma = \{0, \ldots, s\}$, $Q = \{0, \ldots, m\}$, $q_0 = 0$.
- A word $W \colon \omega \to \Sigma$ is a partition $(W_0, \ldots, W_s)$ of ω; $W_j = W^{-1}(j)$.

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Key Observation. The automaton A accepts a word $W\colon \omega \to \Sigma$ iff $\mathcal{P}(\omega), [w_i \mapsto W_i] \models \alpha(w_0, \dots, w_s)$, where α is the $\exists$ $\mathcal{L}$-formula:

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- Describe the automaton A with an existential first order formula φ' in the temporal algebra $\mathcal{P}(\omega)$.
- **Conclusion.** $\mathcal{P}(\omega)$ believes that any first order formula φ is equivalent to an existential formula φ'.

Model companions and languages

Theorem.

The first order theory T^* of an algebra for word languages, $\mathcal{P}(\omega)$,

is the model companion of

a theory T of algebras for a linear temporal logic.

Ghilardi & G. JSL 2017

Model companions and languages

Theorem.

The first order theory T^* of an algebra for tree languages, $\mathcal{P}(2^*)$,

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Ghilardi & G. LICS 2016

Overview

Tree automata

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- The automaton $\mathcal{A}$ accepts (T, R, t_0, σ) if there is a run r such that, for every infinite path $(t_i)_{i=1}^{\omega}$ in T, the number

$$\min\{\Omega(q) \mid r(t_i) = q \text{ for infinitely many } i\}$$

is even.

Rabin-Janin-Walukiewicz theorem

Theorem (Rabin 1969, Janin & Walukiewicz 1995)

For any MSO formula $\Phi(\overline{p})$, there exists an automaton $\mathcal{A}_\Phi$ on the alphabet $\Sigma = 2^{\overline{p}}$ such that, for any tree (T, R, t_0) and colouring $\sigma\colon T \to \Sigma$,

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Here, T_ω is the ω-unravelling of the tree T.

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$$\mathbf{AF}\left(\bigvee_{\Omega(q')<n} q', \neg\left[\bigvee_{\Omega(q)=n} q\right]\right) = \top.$$

Axiomatizing fair CTL

- Boolean algebra axioms,
- Standard axioms for $\Diamond$ and **I**,
- Fixpoint axioms and rules for **EU**:

$$a \vee (b \wedge \Diamond \mathbf{EU}(a, b)) \leq \mathbf{EU}(\varphi, \psi)$$

$$\frac{a \vee (b \wedge \Diamond c) \leq c}{\mathbf{EU}(a, b) \leq c}$$

- and for **EG**:

$$\mathbf{EG}(a, b) \leq a \wedge \Diamond \mathbf{EU}(b \wedge \mathbf{EG}(a, b), a)$$

$$\frac{c \leq a \wedge \Diamond \mathbf{EU}(b \wedge c, a)}{c \rightarrow \mathbf{EG}(a, b)}$$

- **AF** is an abbrevation, $\mathbf{AF}(\varphi, \psi) := \neg \mathbf{EG}(\neg \varphi, \neg \psi)$.

Algebras for fair CTL

- A fair CTL algebra is a tuple $(A, \mathbf{I}, \Diamond, \mathbf{EG}, \mathbf{EU})$ such that
  - A is a Boolean algebra,
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- Given this result, the tree automata from the Rabin-Janin-Walukiewicz theorem can be used to prove that the theory of fair CTL algebras has a model companion.

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$$HA \models \varphi \leq \psi \iff HA \models \varphi^y \leq \psi,$$

$$HA \models \theta \leq \varphi \iff HA \models \theta \leq \varphi_y.$$

- For a different, topological proof of Pitts' theorem, see my paper with Reggio, Top. Appl. 2018.

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- Pitts' operators thus roughly play the role for Heyting algebras that automata played in our results on LTL and CTL.
- Can we say more about this analogy?

Summary of Part II: Model companions

- Model companions are a logical way to think about existentially closed structures; the canonical example is algebraically closed fields.
- Logical algebras, in particular those for linear temporal logic and computation tree logic, admit model companions.
- Automata are crucially used in the proofs, to *eliminate alternations of quantifiers*.
- Heyting algebras have a model companion too, albeit for an (apparently) different reason.

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- Thank you!