Preserving joins at primes

a connection between lattices, domains, and automata

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About this talk

- **1st Objective.** Show an instance of interaction between lattice theory, domain theory and automata.

- **2nd Objective.** Get your feedback on our book:

  Mai Gehrke and Sam van Gool. *Topological duality for distributive lattices, and applications*.
  Preprint, 310pp. arXiv:2203.03286

Overview

Duality for modalities

Duality for implications

Implications that preserve joins at primes

Application 1: domain theory

Application 2: profinite algebra

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The Priestley dual space

- In logic, a ‘complete type’ is a process that gives a consistent yes/no answer to every formula of the logic.
- In the algebraic setting of a bounded distributive lattice L, a complete type is represented by a *lattice homomorphism* $x\colon L \to 2$, or equivalently by a *prime filter* $x^{-1}(\top)$ of L.
- These homomorphisms form a *compact ordered topological space*, X_L, known as the *Priestley dual space* of L.
- Priestley spaces are compact ordered spaces satisfying a strong form of *separation* by clopen cones.
- The lattice L can be recovered from this space X_L as the clopen cones.

Priestley duality

- Algebraic constructions of lattices correspond to topological constructions of Priestley spaces:

$$\begin{aligned} \text{lattice} &\leftrightarrow \text{Priestley space} \\ \text{finite product} &\leftrightarrow \text{disjoint union} \\ \text{quotient lattice} &\leftrightarrow \text{subspace} \\ \text{sublattice} &\leftrightarrow \text{quotient space} \end{aligned}$$

- These correspondences are thanks to a dual equivalence:

DL	$\simeq^{\mathrm{op}}$	**Priestley**
lattice homomorphisms		continuous monotone functions
$L \to M$		$X_M \to X_L$

Priestley and Stone

- Priestley's duality is an alternative view on Stone's original duality
$$\textbf{DL} \simeq^{\mathrm{op}} \textbf{Spec}$$

- Here, **Spec** is a category of topological spaces that have a basis of *compact-open* sets, and maps between such spaces are required to be *spectral*, i.e., inverse image preserves compact-open sets.

- The categories **Spec** and **Priestley** are *isomorphic*.

Modalities

- A *modality* is a function $f : L \to M$ that preserves finite meets.
- A modality can be described by a lattice homomorphism

$$[\![-]\!] : F_\Box(L) \to M,$$

where $F_\Box(L)$ is the lattice of 'free modal terms' over L:

$$F_\Box(L) \stackrel{\text{def}}{=} F_{DL}(\Box L)/\theta$$

and θ is the lattice congruence generated by the pairs $(\Box\top, \top)$ and $(\Box(a \wedge b), \Box a \wedge \Box b)$, for $a, b \in L$.

- $F_\Box$ is the comonad induced by the adjunction $\wedge\mathbf{SLat} \leftrightarrows \mathbf{DL}$.

Duality for modalities

- By duality, a modality $f\colon L \to M$ corresponds to a map $X_M \to \mathcal{V}(X_L)$, where $\mathcal{V}$ is the dual of the functor $F_\Box$.

Proposition

For any Priestley space X, $\mathcal{V}(X)$ is naturally isomorphic to a space of closed up-sets of X, with appropriate topology.

- The space $\mathcal{V}(X)$ is called the *upper Vietoris space* of X.

Corollary

Modalities on L are in bijection with continuous order-preserving functions $X_L \to \mathcal{V}(X_L)$, i.e., compatible binary relations on X_L.

Filters and Vietoris

We sketch a proof of the fact that $\mathcal{V}(X_L)$ is dual to $F_\Box(L)$.

- The points of the dual space of $F_\Box(L)$ are homomorphisms $F_\Box(L) \to 2$.
- Such homomorphisms are in bijection with modalities $L \to 2$.
- Such modalities can be described as filters of L.
- A topological argument shows that filters of L correspond to closed up-sets of X_L.
- The natural topology on $\mathrm{Hom}(F_\Box(L), 2)$ can then be translated to a topology on the closed up-sets of X_L.

Semantics from duality

Calculating a bit further, we recover the core of Kripke semantics:

a modality $\Box \colon L \to M$

is dual to

the relation $R_\Box : X_M \nrightarrow X_L$ defined by:

$$x R_\Box y$$
$$\text{iff}$$
$$\text{for every } a \in L,\ x \in \Box a \text{ implies } y \in a.$$

(we use the notation "$x \in a$" to mean $x(a) = \top$)

Overview

Implication connectives

- In a logic L with at least $\vee$ and $\wedge$, we will say a connective $\Rightarrow$ is an *implication connective* if, for any formulas A, A', B, B',

  1. the formulas $\bot \Rightarrow A$ and $A \Rightarrow \top$ are tautologies of L, and
  2. the following two equivalences hold in L:

$$(A \vee A') \Rightarrow B \equiv (A \Rightarrow B) \wedge (A' \Rightarrow B),$$
$$A \Rightarrow (B \wedge B') \equiv (A \Rightarrow B) \wedge (A \Rightarrow B').$$

Implication operators

A more algebraic formulation of the same concept:

- A binary operation $\Rightarrow$ on a bounded distributive lattice L is an *implication operator* if, for any elements a, a', b, b' of L,

  1. $\bot \Rightarrow a = \top = a \Rightarrow \top$, and
  2. the following two equalities hold in L:

$$(a \vee a') \Rightarrow b = (a \Rightarrow b) \wedge (a' \Rightarrow b),$$
$$a \Rightarrow (b \wedge b') = (a \Rightarrow b) \wedge (a \Rightarrow b').$$

- Note: given an implication operator $\Rightarrow$ on L, for every fixed $a \in L$, the operation $b \mapsto a \Rightarrow b$ is a modality on L.

Implications, a functorial view

- An implication operator on L can be alternatively described by a lattice homomorphism

$$[\![-]\!]\colon F_{\Rightarrow}(L) \to L,$$

  where $F_{\Rightarrow}(L)$ is the lattice of 'free implication terms' over L.

- Formally:

$$F_{\Rightarrow}(L) \stackrel{\text{def}}{=} F_{DL}(L \times L)/\theta_{\Rightarrow},$$

  where $\theta_{\Rightarrow}$ is the congruence generated by the defining equalities for implication.

- Question for the audience: is $F_{\Rightarrow}$ a comonad, like $F_{\Box}$ is?

Duality for implications

- By duality, a lattice homomorphism $[\![-]\!]\colon F_{\Rightarrow}(L) \to L$ corresponds to a map

$$r\colon X_L \to R(X_L)$$

where R is the construction dual to $F_{\Rightarrow}$. But what is this R?

Theorem

The dual space of $F_{\Rightarrow}(L)$ is isomorphic to the space of continuous functions from X_L to $\mathcal{V}(X_L)$ with the compact-to-open topology.

Duality for implications, proof sketch

- For any set X, the dual space of $F_{DL}(X)$ is 2^X.
- Since $F_{\Rightarrow}(L)$ is defined as a quotient of $F_{DL}(L \times L)$, its dual space is a Priestley-closed subspace of $2^{L\times L}$.
- **General methodology.** Let θ be a congruence on L generated by a set of pairs E. Then the dual of L/θ is the closed subspace of points $x \in X_L$ that *verify* all equations in E, i.e. for any $(a, b) \in E$, $x(a) = 1$ iff $x(b) = 1$.
- Applying this method to the equations for $\Rightarrow$, we find a subspace Z of $2^{L\times L}$ consisting of 'filtering' relations on L.
- A topological proof then shows that the space Z is naturally isomorphic to $[X_L, \mathrm{Filt}(L)] \cong [X_L, \mathcal{V}(X_L)]$.

Duality for implications, conclusion

The dual of an implication operator $\Rightarrow$ on L is a function $r\colon X_L \to (X_L \to \mathcal{V}(X_L))$ such that:

- r is spectral, and
- for each $x \in X_L$, $r(x)\colon X_L \to \mathcal{V}(X_L)$ is continuous.

We write $R(X)$ for the space of continuous functions $[X, \mathcal{V}(X)]$.

Semantics from duality for implications

an implication $\Rightarrow : L \times L \to L$

is dual to

the function $r_{\Rightarrow} : X_L \to R(X_L)$ defined by:

$$x r_{\Rightarrow}(z) y$$

iff

for every $a, b \in L$, if $x \in a$ and $z \in a \Rightarrow b$, then $y \in b$.

Overview

Implication and disjunction

- Implication does not need to preserve disjunctions in the second coordinate; i.e., in logical terms,

$$(A \Rightarrow B) \vee (A \Rightarrow C) \text{ is stronger than } A \Rightarrow B \vee C,$$

  and not always equivalent.

- For example, Harrop's formulas

$$(\neg p \to q) \vee (\neg p \to r) \text{ and } \neg p \to (q \vee r)$$

  are not provably equivalent in intuitionistic logic.

Preserving joins at primes

- We say an implication operator $\Rightarrow$ on a bounded distributive lattice L *preserves joins at primes* if

  1. $a \Rightarrow \bot = \bot$ whenever $a \neq \bot$, and
  2. for any prime filter x of L, $a \in x$ and for any $b, c \in L$, there exists $a' \in x$ such that

  $$a \Rightarrow (b \vee c) \leq (a' \Rightarrow b) \vee (a' \Rightarrow c).$$

Preserving joins at primes, canonical extension view

To explain the terminology, we give an equivalent formulation using the *canonical extension* of L, $L^\delta \cong \mathrm{Up}(X_L)$.

- An implication operator $\Rightarrow$ on L *preserves joins at primes* iff for any completely join-prime element x of L^δ, the following function preserves finite joins:

$$\begin{aligned} x \Rightarrow (-) \colon L &\to L^\delta, \\ b &\mapsto \bigvee\{a \Rightarrow b \mid x \leq a \in L\}. \end{aligned}$$

Preserving joins at primes, dually

- Among all implication operators, there are the special ones that *preserve joins at primes*.

- Recall that a general implication operator $\Rightarrow$ on L corresponds dually to a function $r\colon X_L \to R(X_L)$.

- What property of r ensures that $\Rightarrow$ preserves joins at primes?

Functionality

- The space X_L embeds in $\mathcal{V}(X_L)$ by sending any point x to its closed up-set $\uparrow x$, and the image is Priestley-closed.
- The space $[X_L, X_L]$ of continuous functions then also embeds into $R(X_L) = [X_L, \mathcal{V}(X_L)]$, by sending f to $\lambda x. \uparrow f(x)$.
- Denote the image of this embedding by $FR(X_L)$. The subspace $FR(X_L)$ is generally *not* Priestley-closed in $R(X_L)$.

Theorem

Let θ be a congruence on $F_{\Rightarrow}(L)$ and let Z be the corresponding Priestley-closed subspace of $R(X_L)$. Then $\Rightarrow$ preserves joins at primes modulo θ if, and only if, $Z \subseteq FR(X_L)$.

Join-preserving at primes, semantically

for an implication $\Rightarrow : L \times L \to L$,
the property of preserving joins at primes

is dual to

the function $r_{\Rightarrow} : X_L \to R(X_L)$
has the **functionality** property, i.e.,

for every x, z, the set of y such that $xR_{\Rightarrow}(z)y$ has a minimum.

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Denotational semantics

- When writing a program in an (idealized) language, e.g., a λ-calculus, a central question is: what do the programs *mean*?
- One way to attach meaning to programs is to give a compositional interpretation of the language in a category $\mathbf{D}$: *programs* become arrows, and *types* become objects.
- In order to express properties of types, and to model untyped languages, one needs to solve equations such as, e.g.,

$$X \cong [X, X] \cong X \times X.$$

- An idea pursued by D. Scott, Plotkin, and many others: look for $\mathbf{D}$ inside the category of *directedly complete partial orders*.
- A (very) special subcategory: *bifinite domains*.
- This category has a *function space* construct and allows for an incremental solution of equations between domains.

Domain theory in logical form

- A *bifinite domain* is a partially ordered set that is both a limit and a colimit of a directed diagram of its finite 'retracts'.

- **Link with Stone-Priestley duality:**

  Bifinite domains are spectral spaces in their Scott topology.

- In *Domain theory in logical form*, S. Abramsky used this link to analyze the category of bifinite domains via its dual category of bifinite distributive lattices.

Preserving joins at primes in bifinite domains

- For example, to prove that for any bifinite domain X, the domain $[X, X]$ is again bifinite, one may work with the dual lattice of $[X, X]$. What is this dual lattice?
- **Theorem (Abramsky).** Let X be a bifinite domain and L the distributive lattice of compact-open sets of X. Then the space $[X, X]$ is bifinite, and dual to the lattice

$$F_{\Rightarrow}(L)/\theta_j$$

  where θ_j is the congruence generated by the condition that the operator $\Rightarrow$ *preserves joins at primes*.
- *Domain equations* between bifinite domains like $X \cong [X, X]$ may now be solved by lattice-theoretic means.

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Regular languages

- *Regular languages* are collections of finite strings in a finite alphabet Σ that can be described by regular expressions, or equivalently, by finite automata.
- A finite automaton can always be determinized to give a finite *monoid*; regular languages are then the subsets of the free monoid that are saturated under some finite index congruence:

$$L \subseteq \Sigma^* \text{ regular } \iff \exists\ \Sigma^* \overset{f}{\twoheadrightarrow} M \text{ finite monoid s.t. } L = f^{-1}(f(L)).$$

A lattice of regular languages

- Fix a finite alphabet Σ.
- The collection $\mathcal{R}$ of regular languages in Σ forms a *distributive lattice* (in fact, even a Boolean algebra).
- The lattice $\mathcal{R}$ comes equipped with two *implication operators*: for regular languages P, L, and S, the languages

$$P \Rightarrow L \stackrel{def}{=} \{w \in \Sigma^* \mid \forall p \in P, pw \in L\},$$

$$L \Leftarrow S \stackrel{def}{=} \{w \in \Sigma^* \mid \forall s \in S, ws \in L\},$$

  are again regular.
- These two implication operators thus correspond via duality to two functions $r, \ell : X_{\mathcal{R}} \to R(X_{\mathcal{R}})$.

The free profinite monoid as a dual space

Lemma. For any $x, y \in X_{\mathcal{R}}$, each of the sets $\ell(x)(y)$ and $r(y)(x)$ has a unique minimum element, and these two are equal.
Write $x \bullet y := \min \ell(x)(y)$.

Theorem (Gehrke, Grigorieff, Pin 2008)

The Stone space $X_{\mathcal{R}}$ equipped with this operation $\bullet$ is isomorphic to the free profinite monoid over Σ.

Also see V. Moreau's talk in this workshop for more about this monoid!

- **Quiz.** In the first part, we saw that implication operators correspond dually to ternary *relations* $X \to [X \to \mathcal{V}(X)]$. But here we have a binary monoid *operation* $X \to [X \to X]$. Why?

Preserving joins at primes in profinite algebra

- **Answer.** The implication operators in the lattice of regular languages preserve joins at primes!
- Another special property of this setting, which explains why the two functions ℓ and r give a single operation, is that $\Rightarrow$ and $\Leftarrow$ are residual to each other, i.e.,

$$S \subseteq P \Rightarrow L \iff P \subseteq L \Leftarrow S.$$

- The example of regular languages generalizes to:

Theorem (Gehrke 2016)

Topological algebras on a Boolean space X are dual to Boolean algebras equipped with residuated implication operators that preserve joins at primes.

A textbook on duality theory

- If you'd like to learn more about all this...

  Mai Gehrke and Sam van Gool. *Topological duality for distributive lattices, and applications*.

  Preprint (v2), 310pp, May 2022. arXiv:2203.03286
- The first seven chapters are available now and the last chapter (on automata and profinite monoids) will be added soon.
- Any questions or feedback on the draft are *very* welcome at:

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