Model theory and pro-aperiodic monoids

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Joint work with Benjamin Steinberg

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We then exploit existence and uniqueness results about saturated and prime models.

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First-order logic and pro-aperiodic monoids

Pseudofinite words

Saturated models

Prime models

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- Any sentence φ defines a language $L_\varphi := \{w \in \Sigma^* \mid w \models \varphi\}$.
- First Order (FO) logic: disallow second order.

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  any position that has a successor, has an immediate successor.
- True in every finite word, but not in, e.g., $\mathbb{Q}$.

Büchi's theorem

Let $L \subseteq \Sigma^*$ be a language and $M = \Sigma^*/{\approx_L}$ its syntactic monoid.
Then

the monoid M is finite

if, and only if,

there is an MSO-sentence φ such that $L = \{w \in \Sigma^* \mid w \models \varphi\}$.

Schützenberger's theorem

Let $L \subseteq \Sigma^*$ be a language and $M = \Sigma^*/{\approx_L}$ its syntactic monoid. Then

the monoid M is finite aperiodic (i.e., all subgroups are trivial)

if, and only if,

there is an FO-sentence φ such that $L = \{w \in \Sigma^* \mid w \models \varphi\}$.

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- This breaks down for pseudovarieties of finite structures:
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$$\exists n \in \omega \text{ such that the equation } x^n = x^{n+1} \text{ holds.}$$

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- $\to$ Solution: profinite monoids.

Profinite monoids

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  any function $f\colon \Sigma \to M$ has a unique continuous homomorphic extension $\bar{f}\colon \widehat{F}_{\mathbf{V}}(\Sigma) \to M$.

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- The property then also holds with respect to pro-$\mathbf{V}$ monoids M, i.e., inverse limits of finite monoids in $\mathbf{V}$, taken in the category of topological monoids, equivalently Stone spaces equipped with a continuous monoid operation.
- The clopen sets in $\widehat{F}_{\mathbf{V}}(\Sigma)$ are exactly sets of the form $\overline{L}$, for L a language with $M_L \in \mathbf{V}$.

The free pro-aperiodic monoid

Now consider the pseudovariety **A** of aperiodic monoids.

An element u of $\widehat{F}_{\mathbf{A}}(\Sigma)$ can be described as:

- an implicit operation $(u_f)_{f\colon \Sigma \to M \in \mathbf{A}}$,
- an ultrafilter of languages

$$\mathcal{N}_u := \{L \subseteq \Sigma^* \text{ with } M_L \text{ aperiodic, } u \in \overline{L}\},$$

- a complete first-order theory

$$T_u := \{\varphi \text{ first-order sentence} \mid u \in \overline{L_\varphi}\}.$$

- an elementary equivalence class of pseudo-finite words.

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- What is $\mathcal{T}_{fin}$?

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Theorem

The theory $\mathcal{T}_{fin}$ is not finitely axiomatizable.

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Proposition (cf. Doets, 1987)

A word W is pseudofinite if, and only if, for every first-order formula $\varphi(x)$, the set of positions i in W such that $\varphi(i)$ is true has a least element, or is empty.

A monoid of pseudofinite words

- Two pseudofinite words W and W' are elementarily equivalent, notation $W \equiv W'$, if $\mathcal{T}(W) = \mathcal{T}(W')$, i.e., W and W' satisfy exactly the same first-order sentences.

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Theorem (G. & Steinberg)

The topological monoid of elementary equivalence classes of pseudofinite words is the free pro-aperiodic monoid $\widehat{F}_{\mathbf{A}}(\Sigma)$.

Homomorphisms between free pro-aperiodic monoids

- Let Σ, Π be finite alphabets. The continuous homomorphisms $h \colon \widehat{F}_{\mathbf{A}}(\Sigma) \to \widehat{F}_{\mathbf{A}}(\Pi)$ can be described as follows.
- For every $a \in \Sigma$, pick a pseudofinite word W_a in the elementary equivalence class $h(a)$.
- For any element u of $\widehat{F}_{\mathbf{A}}(\Sigma)$, to find the value of $h(u)$, pick a pseudofinite word U in its elementary equivalence class.
- The model
$$U[a/W_a],$$
obtained by substituting for every occurrence of a letter $a \in \Sigma$ the pseudofinite word W_a, is a pseudofinite word in the elementary equivalence class $h(u)$.
- For example, the endomorphism $x \mapsto x^\omega$ can be realized by concatenating a word with itself 'ω times'.

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- However, W_2 contains W_1 as a closed interval; W_3 does not.
- Any closed interval in W_3 realizes all possible factorizations.

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Theorem (Model theory)

Any elementary equivalence class of A-words contains an ω-saturated A-word, which is unique up to isomorphism.

How to use saturated words?

Theorem (G. & Steinberg)

A substitution of ω-saturated words into ω-saturated words is again ω-saturated.

In particular, ω-saturated words are closed under concatenation and ρ-power, where ρ is the ω-saturated order $\mathbb{N} + \mathbb{Q} \times_{\mathrm{lex}} \mathbb{Z} + \mathbb{N}^{\mathrm{op}}$.

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The proof combines a topological characterization of weak saturation with the fact that substitutions are continuous maps between pro-aperiodic monoids.

Equidivisibility

A monoid M is called equidivisible if for any u, v, u', v' in M, $uv = u'v'$ implies that there exists x in M such that $ux = u'$ and $xv' = v$, or $u'x = u$ and $xv = v'$.

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Pick an ω-saturated word W in the class w.

Equidivisibility

A monoid M is called equidivisible if for any u, v, u', v' in M, $uv = u'v'$ implies that there exists x in M such that $ux = u'$ and $xv' = v$, or $u'x = u$ and $xv = v'$.

Proposition

The monoid $\widehat{F}_{\mathbf{A}}(\Sigma)$ is equidivisible.

Proof.

Let $w := uv = u'v'$.

Pick an ω-saturated word W in the class w.

Find x by drawing a picture realizing the two factorizations in W. □

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- They used a normal form due to McCammond to show that if s and t are aperiodic-equivalent then W_s and W_t are isomorphic.
- We can now simply remark that W_s and W_t are ω-saturated, and therefore isomorphic if they are elementarily equivalent, by model theory.

Outline

First-order logic and pro-aperiodic monoids

Pseudofinite words

Saturated models

Prime models

Choosing a model, again

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    1. $W_1 = a^{\mathbb{N}+\mathbb{N}^{\mathrm{op}}}$
    2. $W_2 = a^{\mathbb{N}+\mathbb{Z}+\mathbb{N}^{\mathrm{op}}}$
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- Such a model is called prime.

Prime models

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This, combined with uniqueness of prime models, gives an alternative proof of the fact that the 'cluster words' associated to $u, v \in \widehat{F}_{\mathbf{A}}(\Sigma)$ are isomorphic iff $u = v$.

[1] `https://www.samvangool.net/papers/GS2019primemodels-note.pdf`