Solving temporal equations by folding de Bruijn graphs

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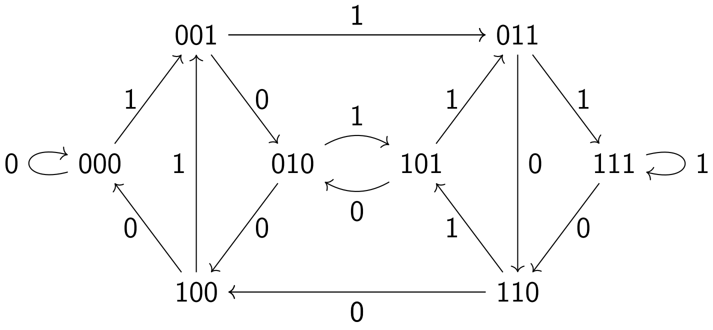

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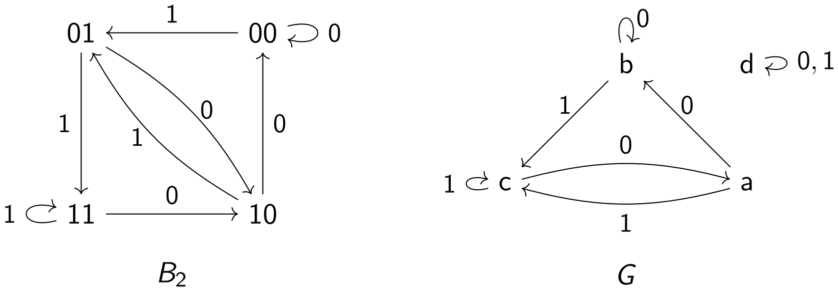

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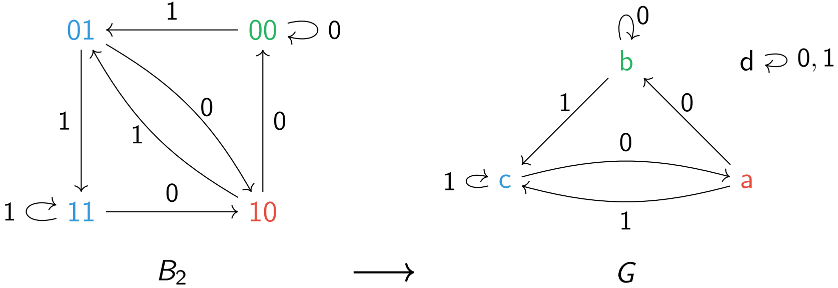

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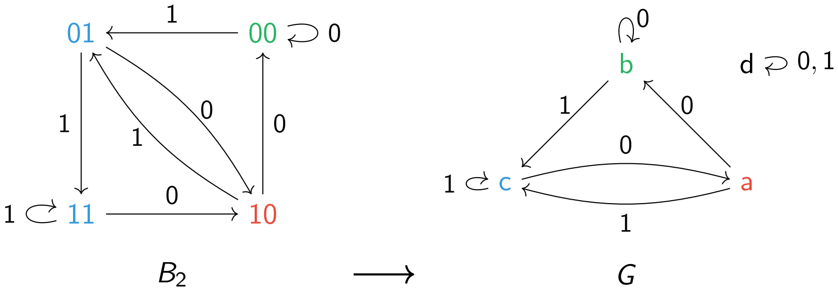

Note:

- The codomain graph may fail to be deterministic.
- There may be more than one homomorphism.

A decision problem

Problem (de Bruijn graph mapping problem)

Given a finite edge-labeled graph $G = (V_G, E_G)$, *do there exist* $d \geq 1$ *and a homomorphism* $B_d(\Sigma) \to G$*?*

Temporal equations

We arrived at this problem because of a problem in temporal logic:

Problem (Unifiability in temporal logic of next)

Given a system of equations in the temporal logic of next X, *does it have a unifier in this logic?*

Unifiers defined

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The depth of a substitution is the maximum nesting of X in the formulas σ_x, for x a variable.

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In particular, decidability of the de Bruijn graph mapping problem implies decidability of the unifiability problem.
(It is in fact equivalent.)

Example of the graph associated to a formula

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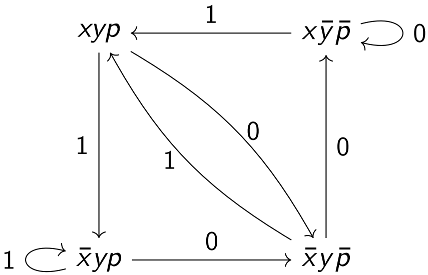

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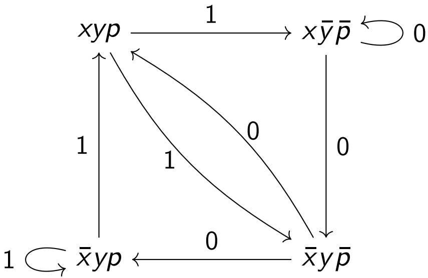

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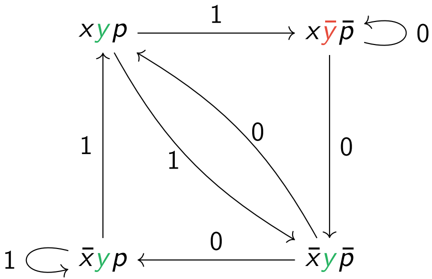

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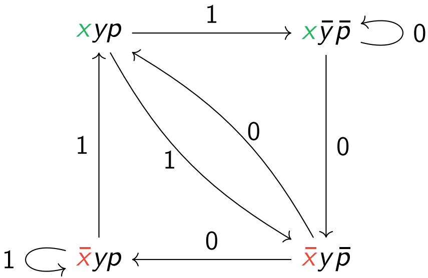

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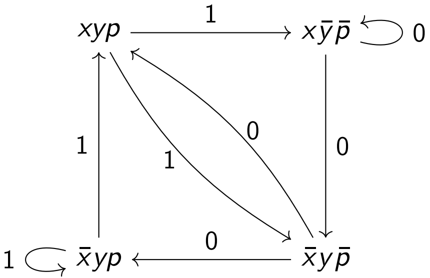

$G(\varphi)$ reversed

The graph $G(\varphi)$ is an image of B_2 (in a unique way).

Deterministic de Bruijn graph images

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Definition

A graph G is an image if there exist $d \geq 1$ and a surjective homomorphism $B_d(\Sigma) \twoheadrightarrow G$.

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- G is d-synchronizing, for $d \geq 1$, if G has a w-sink for every w of length d.

Deciding deterministic images

Proposition (Bleak, Cameron, Maissel, Navas, Olukoya 2016)

Let G be a deterministic graph. Then:

G is an image of B_d

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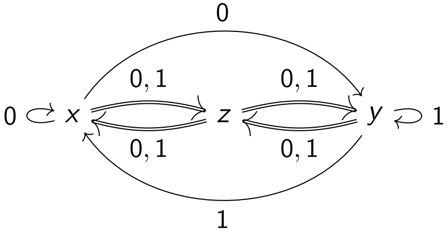
This characterization yields a linear time decision procedure for deterministic input graphs:

1. Set $G_0 := G$.
2. Set $G_{n+1} := G_n/{\equiv_n}$, where $u \equiv_n u'$ iff $u \cdot a = u' \cdot a$ for all a.
3. When $G_{n+1} = G_n$, output 'yes' iff G_n has a single node.

A non-deterministic example: 'The hamburger'



This graph is strongly connected and 2-synchronizing.

However, it is not an image of B_2.

In fact, it is not an image of B_d for any d.

Power-connectedness: intuition

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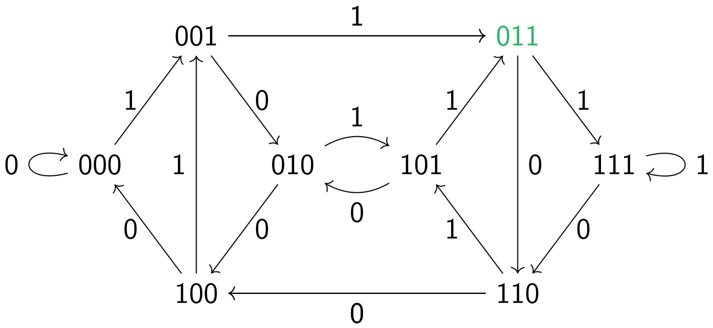

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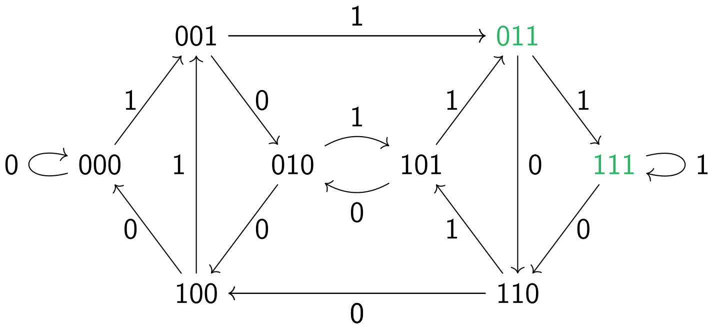

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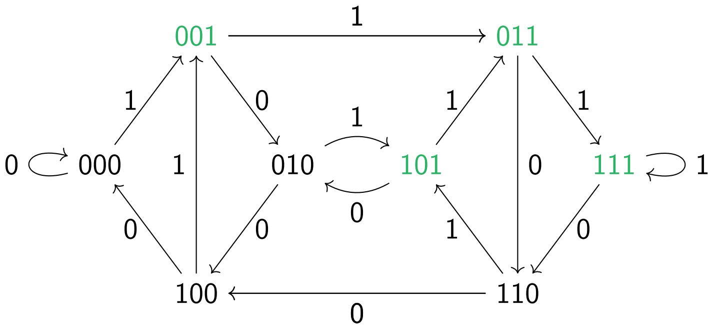

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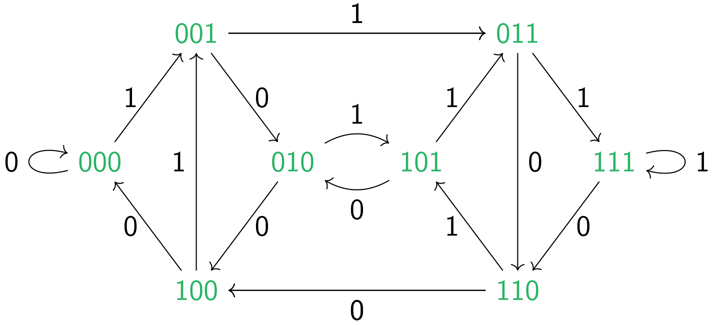

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- G is power-connected if, in its power graph, the node V_G is in the closure of the set of nodes $\{\{u\} \ : \ u \in V_G\}$.

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The hamburger fails to be power-connected, and thus cannot be an image.

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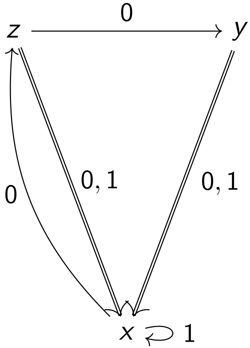

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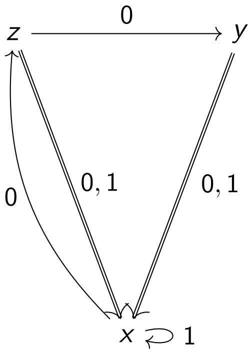

This graph is power-connected, but cannot have a homomorphism from any B_d: There is no self-loop labeled 0.

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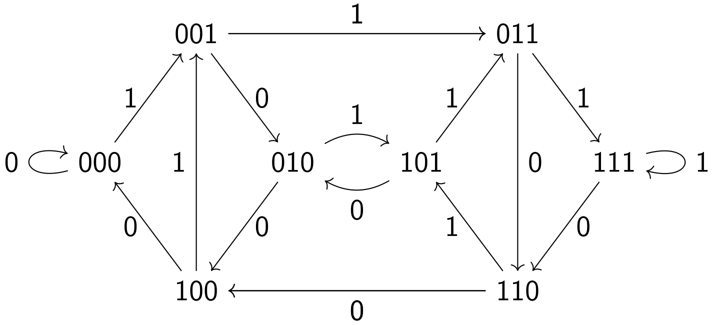

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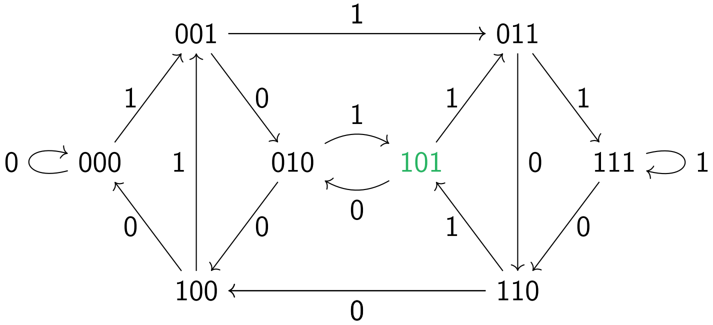

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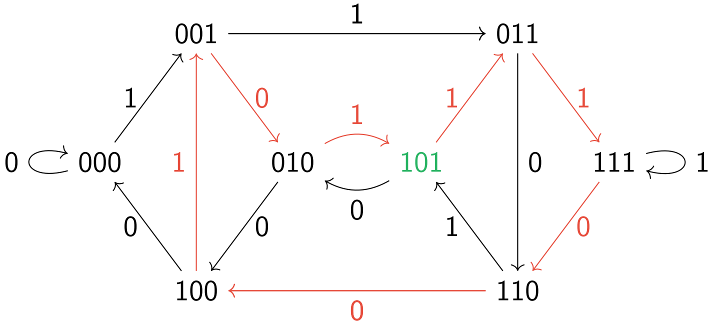

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The two conditions are independent from each other:

- the hamburger is cycle-connected and not power-connected,
- the cone of fries is power-connected and not cycle-connected.

The two conditions can be checked in exponential time.

Proof overview

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- Idea: Use *minimizers*: If w is not highly periodic, single out a position for starting a new synchronization process. (Schleimer et al. 2003, Roberts et al. 2004)

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- From there, a (so far) somewhat *ad hoc* construction of the homomorphism.

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- A connection to symbolic dynamics: Our main result implies that *it is decidable if a two-sided edge shift has a sliding block code section* (One-sided case: Salo and Törmä 2015.)
- The general method is not limited, in principle, to de Bruijn graphs and temporal logic of next. One might try to use it for unifiability for other logics, e.g., modal logic **K** (hypergraphs!)

Final remarks

- We establish that the mapping problem is in exp-time, and the unifiability problem in 2-exp-time. Hardness?
- A connection to symbolic dynamics: Our main result implies that *it is decidable if a two-sided edge shift has a sliding block code section* (One-sided case: Salo and Törmä 2015.)
- The general method is not limited, in principle, to de Bruijn graphs and temporal logic of next. One might try to use it for unifiability for other logics, e.g., modal logic **K** (hypergraphs!)
- Thanks for your attention! (Who else is ready for lunch...?)