

# Modal logic

Sam van Gool

April 9, 2026

These are lecture notes for the part of the L3 course *Logique* at ENS Paris-Saclay in which we talk about modal logic. Sections 1–3 are based in part on the first two chapters of the book [1]. Sections 4 and 5 are based on [2, Sec. 4.4].

## Contents

1	Syntax and semantics	1
2	A Hilbert system	2
3	Bisimulation-invariance	5
4	Completeness	6
5	Stone-Jónsson-Tarski representation	8

## 1 Syntax and semantics

**Definition 1.** We fix a set of propositional variables  $P$ . A *formula* of modal propositional logic is an expression generated by the following grammar:

$$\varphi ::= \top \mid p \mid \varphi \vee \varphi \mid \neg\varphi \mid \Box\varphi .$$

We call a formula *Boolean* if it contains no  $\Box$ .

**Remark 2.** The study of modal logic in fact covers a more general syntax, where one can have several modalities, instead of just one, and the modalities can be  $n$ -ary for  $n > 1$ , instead of unary. We do not treat this here. It is also possible to work over a *positive* instead of a Boolean base, meaning that one takes  $\vee, \wedge, \top, \perp$  as basic non-modal connectives. We also do not treat this here.

Recall that a Boolean formula is *valid* if it evaluates to 1 under all two-valued valuations of propositions. We extend this notion of validity to modal formulas.

**Definition 3.** A *frame* is a pair  $(W, R)$  where  $W$  is a set and  $R \subseteq W \times W$  is a binary relation.

A *model* is a triple  $(W, R, V)$ , where  $(W, R)$  is a frame and  $V: W \rightarrow \mathcal{P}(P)$  is a function.

A *pointed model* is a quadruple  $(W, R, V, w)$  where  $(W, R, V)$  is a model and  $w \in W$ .

We define the satisfaction relation  $\models$ . Let  $\mathcal{M} = (W, R, V)$  be a model and  $w \in W$ . We define:

- $\mathcal{M}, w \models \top$  always;
- $\mathcal{M}, w \models p$  if, and only if,  $p \in V(w)$ ;
- $\mathcal{M}, w \models \varphi \vee \psi$  if, and only if,  $\mathcal{M}, w \models \varphi$  or  $\mathcal{M}, w \models \psi$ ;
- $\mathcal{M}, w \models \neg\varphi$  if, and only if, it is not the case that  $\mathcal{M}, w \models \varphi$ ;
- $\mathcal{M}, w \models \Box\varphi$  if, and only if, for all  $v \in W$  such that  $wRv$ ,  $\mathcal{M}, v \models \varphi$ .

We also write  $\mathcal{M} \models \varphi$  if, for all  $w \in W$ , we have  $\mathcal{M}, w \models \varphi$ .

For a frame  $\mathcal{F} = (W, R)$ , we write  $\mathcal{F} \models \varphi$  if, for all  $V: W \rightarrow \mathcal{P}(P)$ ,  $(W, R, V) \models \varphi$ .

**Example 4.** Let  $K$  be the formula

$$\Box(p \rightarrow q) \rightarrow (\Box p \rightarrow \Box q).$$

We show that  $\mathcal{M} \models K$  for any model  $\mathcal{M}$ . Let  $w$  be an arbitrary node in  $\mathcal{M}$  and suppose that  $\mathcal{M}, w \models \Box(p \rightarrow q)$  and  $\mathcal{M}, w \models \Box p$ . Let  $v$  in  $\mathcal{M}$  be arbitrary such that  $wRv$ . Then  $\mathcal{M}, v \models p \rightarrow q$  and  $\mathcal{M}, v \models p$ . Thus,  $\mathcal{M}, v \models q$ , as required.

**Definition 5.** We introduce an abbreviation:

$$\Diamond\varphi := \neg\Box\neg\varphi.$$

We also use the usual propositional abbreviations:  $\varphi \wedge \psi$  for  $\neg(\neg\varphi \vee \neg\psi)$ ,  $\varphi \rightarrow \psi$  for  $\neg\varphi \vee \psi$ ,  $\varphi \leftrightarrow \psi$  for  $(\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)$  and  $\perp$  for  $\neg\top$ .

**Remark 6.** We have

$$\mathcal{M}, w \models \Diamond\varphi \text{ if, and only if, there exists } v \in W \text{ such that } wRv \text{ and } \mathcal{M}, v \models \varphi.$$

**Remark 7.** Everything we do here is over a *classical* base. Intuitionistic versions of modal logic exist and this is a currently fairly popular object of study, but we do not treat it in this course.

**Definition 8.** *Local semantic consequence* is defined as follows. Let  $\Gamma$  be a set of formulas and  $\varphi$  a formula. We write  $\Gamma \models \varphi$  if, and only if, for all pointed models  $(\mathcal{M}, w)$ , if  $\mathcal{M}, w \models \gamma$  for all  $\gamma \in \Gamma$ , then  $\mathcal{M}, w \models \varphi$ .

*Global semantic consequence* is the following variant: we write  $\Gamma \models^g \varphi$  if, and only if, for all models  $\mathcal{M}$ , if  $\mathcal{M} \models \gamma$  for all  $\gamma \in \Gamma$ , then  $\mathcal{M} \models \varphi$ .

**Remark 9.** If  $\Gamma \models \varphi$ , then  $\Gamma \models^g \varphi$ .

**Example 10.** We have  $p \models^g \Box p$  but it is not the case that  $p \models \Box p$ .

## 2 A Hilbert system

**Definition 11.** A *normal modal logic* is a set  $\Lambda$  of formulas such that

1. for any Boolean formula  $\varphi$ , if  $\varphi$  is valid then  $\varphi$  is in  $\Lambda$ ;
2. the formula  $K$  is in  $\Lambda$ ;

3. if  $\varphi$  and  $\varphi \rightarrow \psi$  are in  $\Lambda$ , then  $\psi$  is in  $\Lambda$ ;
4. if  $\varphi$  is in  $\Lambda$ , then  $\Box\varphi$  is in  $\Lambda$ ;
5. if  $\varphi(p_1, \dots, p_n)$  is in  $\Lambda$  and  $\sigma_1, \dots, \sigma_n$  are any formulas, then  $\varphi(\sigma_1, \dots, \sigma_n)$  is in  $\Lambda$ .

In other words, a normal modal logic is one which contains (1) **CPC**, (2) the axiom **K**, and is closed under (3) modus ponens, (4) necessitation, and (5) uniform substitution.

The *smallest normal modal logic* is called **K**.

Note that **K** is well-defined because normal modal logics are closed under intersections.

**Definition 12.** Let  $\varphi$  be a formula. A **K**-proof is a non-empty finite sequence  $\varphi_0, \dots, \varphi_n$  such that the following hold:

- $\varphi_n = \varphi$ , and
- for each  $0 \leq i \leq n$ ,  $\varphi_i$  is either a valid Boolean formula, or the axiom **K**, or obtainable from formulas in  $\{\varphi_j \mid j < i\}$  by applying modus ponens, necessitation, or uniform substitution.

**Proposition 13.** Let  $\varphi$  be a formula. Then  $\varphi$  is in **K** if, and only if, there exists a **K**-proof of  $\varphi$ .

*Proof.* Denote by **K'** the set of formulas  $\varphi$  for which a **K**-proof exists. Note that **K'** is a normal modal logic, and thus contains **K**. On the other hand, if  $\Lambda$  is any normal modal logic, then one shows by induction that, for any  $n \in \mathbb{N}$ , if  $\varphi$  admits a **K**-proof of length  $\leq n$ , then  $\varphi \in \Lambda$ . In particular, taking  $\Lambda := \mathbf{K}$ , we get that  $\mathbf{K}' \subseteq \mathbf{K}$ .  $\square$

**Definition 14.** We write  $\vdash_{\mathbf{K}} \varphi$  if the conditions in Proposition 13 hold for  $\varphi$ .

**Remark 15.** The formula  $(\Box p \wedge \Box q) \leftrightarrow \Box(p \wedge q)$  is in **K**. (*Exercise.*)

**Definition 16.** Let  $\varphi$  and  $\psi$  be formulas. We say that  $\varphi$  *implies*  $\psi$  (in **K**) if  $\vdash_{\mathbf{K}} \varphi \rightarrow \psi$ ; we write  $\varphi \preceq_{\mathbf{K}} \psi$  or also just  $\varphi \preceq \psi$ . We say that  $\varphi$  and  $\psi$  are *equivalent* (in **K**) if  $\vdash_{\mathbf{K}} \varphi \leftrightarrow \psi$ . In this case, we write  $\varphi \equiv_{\mathbf{K}} \psi$  or also just  $\varphi \equiv \psi$ .

**Remark 17.** If we would have a **K**-proof of  $p$ , then by necessitation we would also have a **K**-proof of  $\Box p$ . However, we cannot reason as in natural deduction and ‘introduce an arrow’ to obtain from this reasoning a **K**-proof of  $p \rightarrow \Box p$ . In fact, no **K**-proof of  $p$  should exist, because then by the uniform substitution rule we would also have a **K**-proof of  $\perp$ . It will follow from soundness (Theorem 19) that no such **K**-proof exists.

**Remark 18.** Definition 12 is sometimes called a *Hilbert-style proof system*, as opposed to the Gentzen-style systems of natural deduction and **LK**, **LJ**, which we saw earlier in this course. The advantages of a Hilbert-style system are that it is short to define, easy to remember, and that the definition of the notion of ‘proof’ in such a system only requires a notion of ‘list’ in the meta-theory, rather than trees, which can simplify some induction arguments. We will see below that the connection with algebra is also easily made. On the other hand, it is not clear in general how to obtain decidability of the membership problem for a logic when it is given by a Hilbert-style system.

**Theorem 19** (Soundness). Let  $\varphi$  be a formula. If  $\vdash_{\mathbf{K}} \varphi$ , then, for any frame  $\mathcal{F}$ , we have  $\mathcal{F} \models \varphi$ .

*Proof.* We show by induction on the length of a proof of  $\varphi$  that, for any pointed model  $\mathcal{M}, w$ , we have  $\mathcal{M}, w \models \varphi$ .

Let  $n \geq 0$  and suppose that  $\pi = (\varphi_0, \dots, \varphi_n)$  is a **K**-proof of  $\varphi$ , so  $\varphi_n = \varphi$ . Let  $\mathcal{M}, w$  be a pointed model. We distinguish cases according to the justification for  $\varphi_n$  in the proof  $\pi$ .

If  $\varphi_n$  is a valid Boolean formula, then we certainly have  $\mathcal{M}, w \models \varphi$ .

If  $\varphi_n$  is the axiom **K**, then  $\mathcal{M}, w \models \varphi_n$  by Example 4.

Otherwise,  $\varphi_n$  is obtainable by one of the three rules. (In particular, we must then have  $n > 0$ .) Suppose, for instance, that  $\varphi_n$  is obtained by the necessitation rule from  $\varphi_i$  for some  $0 \leq i < n$ , so that  $\varphi_n = \Box\varphi_i$ . The length  $i + 1$  prefix of  $\pi$  is a **K**-proof of  $\varphi_i$ , so the induction hypothesis in particular yields that  $\mathcal{M}, v \models \varphi_i$  for all  $R$ -successors  $v$  of  $w$ . Thus,  $\mathcal{M}, w \models \Box\varphi_i$ , which is  $\varphi_n$ .

The proofs for the cases of modus ponens and uniform substitution are left as an [exercise](#).  $\square$

**Definition 20.** Let  $\Gamma$  be a set of formulas. The smallest normal modal logic that contains  $\Gamma$  is called the *normal modal logic generated by  $\Gamma$*  or *the extension of **K** by  $\Gamma$* . We denote it by **K $\Gamma$**  (with some exceptions, see below).

**Remark 21.** The Hilbert-style definition of **K**-proof can be extended to a definition of **K $\Gamma$** -proof by allowing in addition that a proof contains any element of  $\Gamma$ . The same proof as the one given above for Proposition 13 then shows that the formulas in **K $\Gamma$**  are exactly those for which a **K $\Gamma$** -proof exists. ([Exercise](#).)

**Example 22.** We list a few formulas that are commonly used as axioms, and their commonly used names:

- (4)  $\Box p \rightarrow \Box\Box p$
- (T)  $\Box p \rightarrow p$
- (B)  $p \rightarrow \Box\Diamond p$
- (L)  $\Box(\Box p \rightarrow p) \rightarrow \Box p$
- (D)  $\Box p \rightarrow \Diamond p$

We write **K4** instead of **K**{4}, etc. **K**{4, T} is called **S4** and **K**{4, T, B} is called **S5**. **KL** is also known as **GL**. We do not comment on the history or idiosyncrasy of these names.

**Remark 23.** Some references, like [1], give  $\Diamond$ -versions of some of the axioms. It is a matter of Boolean calculation to translate between the two versions. We do the calculation just for (T):

$$\Box p \rightarrow p \equiv \neg\Diamond\neg p \rightarrow p \equiv \neg p \rightarrow \Diamond\neg p$$

and by applying the substitution  $\neg p \mapsto p$ , we get the diamond version of (T):

$$p \rightarrow \Diamond p .$$

It is an instructive [exercise](#) to translate (L) into its  $\Diamond$ -form. Note that we use in this remark that  $\equiv$  is an equivalence relation, which we prove below (Lemma 38).

**Remark 24.** **KD** can alternatively be axiomatized by the formula  $\Diamond\top$ . ([Exercise](#).)

### 3 Bisimulation-invariance

**Definition 25.** A *bisimulation* from a model  $\mathcal{M}_1 = (W_1, R_1, V_1)$  to a model  $\mathcal{M}_2 = (W_2, R_2, V_2)$  is a relation  $Z \subseteq W_1 \times W_2$  such that the following hold for any  $(w_1, w_2) \in Z$ :

1.  $V_1(w_1) = V_2(w_2)$ ;
2. for any  $u \in W_1$ , if  $w_1 R_1 u$ , then there exists  $v \in W_2$  such that  $w_2 R_2 v$  and  $(u, v) \in Z$ ;
3. for any  $v \in W_2$ , if  $w_2 R_2 v$ , then there exists  $u \in W_1$  such that  $u R_1 v$  and  $(u, v) \in Z$ .

For pointed models, we say that  $\mathcal{M}_1, w_1$  is *bisimilar* to  $\mathcal{M}_2, w_2$  if there exists a bisimulation from  $\mathcal{M}_1$  to  $\mathcal{M}_2$  which contains  $(w_1, w_2)$ .

**Proposition 26.** *Modal formulas are invariant under bisimulation. That is, if  $\mathcal{M}_1, w_1$  is bisimilar to  $\mathcal{M}_2, w_2$ , then for any modal formula  $\varphi$ , we have  $\mathcal{M}_1, w_1 \models \varphi$  if, and only if,  $\mathcal{M}_2, w_2 \models \varphi$ .*

*Proof.* By induction on  $\varphi$ . The case of propositional variables holds by (1) in Definition 25, and the cases  $\vee$  and  $\neg$  are simple. For the case  $\Box$ , suppose that  $\mathcal{M}_1, w_1 \models \Box\psi$ . In order to show that  $\mathcal{M}_2, w_2 \models \Box\psi$ , let  $v \in W_2$  be arbitrary such that  $w_2 R_2 v$ . By (3) in Definition 25, pick  $u \in W_1$  such that  $w_1 R_1 u$  and  $(u, v) \in Z$ . Since  $\mathcal{M}_1, w_1 \models \Box\psi$ , we have  $\mathcal{M}_1, u \models \psi$ . By the induction hypothesis,  $\mathcal{M}_2, v \models \psi$ . The proof of the converse direction is the same, this time using (2) in Definition 25.  $\square$

**Definition 27.** Let  $(\mathcal{M}, w)$  be a pointed model. The *modal theory* of  $(\mathcal{M}, w)$  is the set of modal formulas  $\varphi$  such that  $\mathcal{M}, w \models \varphi$ .

**Definition 28.** A frame  $\mathcal{F} = (W, R)$  is *image-finite* if, for every  $w \in W$ , the relational image  $R[w] := \{v \in W \mid w R v\}$  is a finite set. A model  $(W, R, V)$  is image-finite if its underlying frame  $(W, R)$  is.

**Theorem 29** (Hennessy-Milner). *If  $\mathcal{M}_1$  and  $\mathcal{M}_2$  are image-finite models and  $w_1$  in  $\mathcal{M}_1$  has the same modal theory as  $w_2$  in  $\mathcal{M}_2$ , then  $\mathcal{M}_1, w_1$  is bisimilar to  $\mathcal{M}_2, w_2$ .*

*Proof.* Define  $Z := \{(w_1, w_2) \in W_1 \times W_2 \mid \mathcal{M}, w_1 \text{ and } \mathcal{M}, w_2 \text{ have the same modal theory}\}$ . It suffices to show that  $Z$  is a bisimulation. Condition (1) of Definition 25 is clear. For condition (2), suppose that  $(w_1, w_2) \in Z$  and  $w_1 R_1 u$ . Write  $\{v_1, \dots, v_n\}$  for the (by assumption finite) set of  $R_2$ -successors of  $w_2$ . Towards a contradiction, suppose that none of them has the same modal theory as  $u$ . For each  $1 \leq i \leq n$ , pick a formula  $\varphi_i$  such that  $\mathcal{M}_1, u \models \varphi_i$  and  $\mathcal{M}_2, v_i \not\models \varphi_i$ . Then  $\mathcal{M}_1, w_1 \models \Diamond(\bigwedge_{i=1}^n \varphi_i)$ . Since  $(w_1, w_2) \in Z$ , we also get  $\mathcal{M}_2, w_2 \models \Diamond(\bigwedge_{i=1}^n \varphi_i)$ . This now leads to the desired contradiction. The proof of condition (3) is the same, this time using that  $\mathcal{M}_1$  is image-finite.  $\square$

**Example 30.** It is not the case that, if  $\mathcal{M}_1, w_1$  and  $\mathcal{M}_2, w_2$  have the same modal theory, then there must exist a bisimulation linking  $w_1$  to  $w_2$ . We give a counterexample. Consider  $\mathcal{M}_1$  based on  $W_1 := \{w_1\} \sqcup \bigsqcup_{n \in \mathbb{N}} \{0, \dots, n\}$ . For  $n \in \mathbb{N}$  and  $k \leq n$ , we write  $(n, k)$  for the node  $k$  at component  $n$ . For each  $n \in \mathbb{N}$ , we let  $(n, 0)$  be an  $R_1$ -successor of  $w_1$ , and we let  $(n, i+1)$  be an  $R_1$ -successor of  $(n, i)$  for all  $i < n$ . (Draw a picture.)

Let  $\mathcal{M}_2$  be based on  $\{w_2\} \sqcup \bigsqcup_{n \in \mathbb{N}} \{0, \dots, n\} \sqcup \mathbb{N}$ ; we write  $(\infty, k)$  for elements of the last component, for any  $k \in \mathbb{N}$ . The relation  $R_2$  is defined as  $R_1$ , replacing  $w_1$  by  $w_2$ , and adding  $(\infty, 0)$  as an

$R_2$ -successor of  $w_2$ , and also adding  $(\infty, k+1)$  as an  $R_2$ -successor of  $(\infty, k)$ , for every  $k \in \mathbb{N}$ . (Also draw a picture.)

In both models, we put  $V(w) := \emptyset$  for all worlds  $w$ .

A simple induction shows that for any  $\varphi$ ,  $\mathcal{M}_1, w_1 \models \varphi$  if, and only if,  $\mathcal{M}_2, w_2 \models \varphi$ . However,  $\mathcal{M}_1, w_1$  is not bisimilar to  $\mathcal{M}_2, w_2$ . Indeed, such a bisimulation would need to relate the node  $(\infty, 0)$  in  $W_2$ , being a successor of  $w_1$  to some node  $(n, k)$  in  $W_1$ . But then taking  $n - k$  steps in  $W_1$  yields that  $(n, n)$  is related by the bisimulation to  $(\infty, n)$ . However,  $(n, n)$  has no  $R_1$ -successor but  $(\infty, n)$  does have an  $R_2$ -successor, so the bisimulation is not allowed to link them.

**Remark 31.** There exists a version of Theorem 29 that works more generally for models that are not image-finite, using ultrafilter extensions. See [1, Section 2.5].

**Remark 32.** Modal logic can express precisely the same properties of frames as bisimulation-invariant first-order logic. See also [1, Section 2.6].

## 4 Completeness

We saw in Theorem 19 that  $\mathbf{K}$ -provability implies semantic validity. In this section, we prove the converse (Theorem 33).

**Theorem 33.** *Let  $\varphi$  be a formula. If, for any frame  $\mathcal{F}$ , we have  $\mathcal{F} \models \varphi$ , then  $\vdash_{\mathbf{K}} \varphi$ .*

We use an algebraically inspired approach. This was in fact historically how Theorem 33 was first proved by Jónsson and Tarski (1951), although they presented their results as pure algebra, and the connection with logic was further clarified later. See Section 1.7 of [1] for more on the history.

**Definition 34.** A *Boolean algebra* is a Heyting algebra  $A$  such that, for all  $a \in A$ ,  $a \vee \neg a = \top$ . A *Boolean algebra homomorphism* is a function  $h: A \rightarrow B$  between Boolean algebras which preserves  $\neg, \vee$  and  $\top$ .

A *modal algebra* is a pair  $(A, \Box)$ , where  $A$  is a Boolean algebra, and  $\Box: A \rightarrow A$  is a function that preserves  $\wedge$  and  $\top$ .

**Example 35.** The two-element set  $2 = \{0, 1\}$  is a Boolean algebra with its natural order. More generally, for any set  $W$ , the power set  $\mathcal{P}(W)$  is a Boolean algebra, with the inclusion order. If  $R$  is a binary relation on a set  $W$ , then  $\Box: \mathcal{P}(W) \rightarrow \mathcal{P}(W)$  defined, for  $U \subseteq W$ , by

$$\Box U := \{w \in W \mid \text{for all } v \in W, \text{ if } wRv \text{ then } v \in U\}$$

is a function that preserves (any) intersections, and thus  $(\mathcal{P}(W), \Box)$  is a modal algebra.

Recall the relation  $\preceq$  on formulas, which we defined in Definition 16.

**Lemma 36.** *The relation  $\preceq$  is reflexive and transitive, i.e., a preorder.*

*Proof.* Let  $\varphi$  be a formula. Note that  $p \rightarrow p$  is a valid Boolean formula, and  $\varphi \rightarrow \varphi$  can be obtained from it by substitution, so  $\vdash_{\mathbf{K}} \varphi \rightarrow \varphi$ .

Now let  $\psi, \theta$  also be formulas and suppose that  $\vdash_{\mathbf{K}} \varphi \rightarrow \psi$  and  $\vdash_{\mathbf{K}} \psi \rightarrow \theta$ . Since

$$(p \rightarrow q) \rightarrow ((q \rightarrow r) \rightarrow (p \rightarrow r))$$

is a valid Boolean formula, its substitution instance

$$(\varphi \rightarrow \psi) \rightarrow ((\psi \rightarrow \theta) \rightarrow (\varphi \rightarrow \theta))$$

is  $\mathbf{K}$ -provable. Now apply modus ponens twice to get  $\vdash_{\mathbf{K}} \varphi \rightarrow \theta$ .  $\square$

**Remark 37.** Note that  $\varphi \equiv \psi$  if, and only if,  $\varphi \preceq \psi$  and  $\psi \preceq \varphi$ . ([Exercise.](#))

**Lemma 38.** *The relation  $\equiv$  is an equivalence relation.*

*Proof.* Immediate from Lemma 36 and Remark 37.  $\square$

**Definition 39.** The *Lindenbaum algebra* of modal logic is based on the set  $\mathcal{L}$  of  $\equiv$ -equivalence classes of formulas. We denote by  $[\varphi]$  the equivalence class of  $\varphi$ . We equip  $\mathcal{L}$  with the partial order  $[\varphi] \leq [\psi]$  iff  $\varphi \preceq \psi$ , and with the operation  $\Box: \mathcal{L} \rightarrow \mathcal{L}$  defined by  $\Box[\varphi] := [\Box\varphi]$ .

**Proposition 40.** *The Lindenbaum algebra  $\mathcal{L}$  is a well-defined modal algebra.*

*Proof.* The well-definedness of  $\leq$  is immediate using Lemma 36 and Remark 37. For the well-definedness of  $\Box$ , it suffices to show that, for any formulas  $\varphi, \psi$ ,

$$\text{if } \vdash_{\mathbf{K}} \varphi \rightarrow \psi \text{ then } \vdash_{\mathbf{K}} \Box\varphi \rightarrow \Box\psi .$$

Indeed, from  $\vdash_{\mathbf{K}} \varphi \rightarrow \psi$ , by necessitation, we get  $\vdash_{\mathbf{K}} \Box(\varphi \rightarrow \psi)$ . By substituting  $\varphi$  for  $p$  and  $\psi$  for  $q$  in the axiom  $\mathbf{K}$ , we have  $\vdash_{\mathbf{K}} \Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$ . Thus,  $\vdash_{\mathbf{K}} \Box\varphi \rightarrow \Box\psi$  now follows by modus ponens.

That  $(\mathcal{L}, \leq)$  is a Boolean algebra follows from the fact that all valid Boolean formulas are axioms of  $\vdash_{\mathbf{K}}$ ; in particular,  $[\top]$  is the maximum element of this Boolean algebra, as  $\vdash_{\mathbf{K}} \varphi \rightarrow \top$  for any formula  $\varphi$ . The fact that the operation  $\Box$  on  $\mathbb{F}$  preserves  $\wedge$  follows from Remark 15. Finally, since  $\vdash_{\mathbf{K}} \top$ , necessitation gives  $\vdash_{\mathbf{K}} \Box\top$ , so  $\Box[\top] = [\Box\top] = [\top]$ .  $\square$

**Remark 41.** The Lindenbaum algebra  $\mathcal{L}$  is *the free modal algebra* on set of generators  $\mathbf{P}$ . This means that, for any modal algebra  $(A, \Box)$  and any function  $v: \mathbf{P} \rightarrow A$ , there exists a unique modal algebra homomorphism  $\bar{v}: \mathcal{L} \rightarrow (A, \Box)$  such that  $\bar{v}([p]) = v(p)$  for all  $p \in \mathbf{P}$ . We will only use this fact once, to explain a different viewpoint on a proof below (see Remark 44).

The crucial fact needed for the proof of Theorem 33 is the following.

**Theorem 42** (Jónsson-Tarski 1951). *Let  $(A, \Box)$  be a modal algebra. There exist a frame  $(W, R)$  and an injective Boolean algebra homomorphism  $\eta: A \rightarrow \mathcal{P}(W)$  such that, for any  $a \in A$ , we have*

$$\eta(\Box a) = \{w \in W \mid \text{for all } v \in W, \text{ if } wRv \text{ then } v \in \eta(a)\} . \quad (1)$$

**Remark 43.** When  $(W, R)$  is a frame and  $w \in W$ , write  $R[w]$  for the set of  $R$ -successors of  $w$ . Then (1) says that, for any  $w \in W$ , we have  $\Box a \in w$  if, and only if,  $R[w] \subseteq \eta(a)$ .

We prove the Jónsson-Tarski theorem in the next section (Proposition 54). This is where the real algebraic work will happen. To motivate ourselves, we first show how it yields a proof of completeness.

*Proof that Theorem 42 implies Theorem 33.* Let  $\varphi$  be a formula such that  $\mathcal{F} \models \varphi$  for all frames  $\mathcal{F}$ . Apply Theorem 42 to the Lindenbaum algebra  $(\mathcal{L}, \sqsubseteq)$  of Definition 39, which is a modal algebra by Proposition 40. We obtain a frame  $(W, R)$  and an injective Boolean homomorphism  $\eta: \mathbb{F} \rightarrow \mathcal{P}(W)$  satisfying property (1). Consider the valuation  $V: W \rightarrow \mathcal{P}(\mathbf{P})$  defined, for  $w \in W$ , by

$$V(w) := \{p \in \mathbf{P} \mid w \in \eta(p)\} .$$

We show by induction that, for any formula  $\psi$ , we have

$$\eta([\psi]) = \{w \in W \mid \mathcal{M}, w \models \psi\} . \quad (2)$$

The base case follows from the definition of  $V$ . The case for  $\Box$  uses the equality (1):

$$\begin{aligned} \eta([\Box\psi]) &= \eta([\Box[\psi]]) && \text{(definition of } \Box) \\ &= \{w \in W \mid R[w] \subseteq \eta([\psi])\} && \text{(by (1))} \\ &= \{w \in W \mid \text{for all } v \in R[w], \mathcal{M}, v \models \psi\} && \text{(induction hypothesis)} \\ &= \{w \in W \mid \mathcal{M}, w \models \Box\psi\} && \text{(definition of } \models) . \end{aligned}$$

The cases  $\top, \vee, \neg$  are similar, using that  $\eta$  is a Boolean algebra homomorphism. Applying (2) to the formula  $\varphi$ , we have  $\eta([\varphi]) = W = \top_{\mathcal{P}(W)}$ . Since  $\eta$  is injective, we get  $[\varphi] = [\top]$ , so  $\vdash_{\mathbf{K}} \varphi$ .  $\square$

**Remark 44.** An alternative proof of (2) is: both the function  $[\psi] \mapsto \eta([\psi])$  and the function  $[\psi] \mapsto \{w \in W \mid \mathcal{M}, w \models \psi\}$  are modal algebra homomorphisms, and they are equal on the set of generators  $\{[p] \mid p \in \mathbf{P}\}$ . The two functions must therefore be identical by Remark 41.

## 5 Stone-Jónsson-Tarski representation

We will split up the task of proving Theorem 42 into two steps:

1. Define the set  $W_A$  and the injective function  $\eta$  for the Boolean algebra  $A$  (Definition 50 and Proposition 51);
2. Define an appropriate binary relation  $R$  on  $W$  that satisfies (1) (Definition 52 and Proposition 54).

Step 1 was done by Stone in the 1930s and step 2 was done by Jónsson and Tarski in the 1950s.

**Definition 45.** Let  $A$  be a Boolean algebra. An *filter* in  $A$  is a subset  $F$  of  $A$  such that (1)  $F$  is *upward closed*: for any  $a, b \in A$ , if  $a \in F$  and  $a \leq b$ , then  $b \in F$ ; (2)  $F$  is *closed under finite meets*:  $\top \in F$  and, for any  $a, b \in F$ ,  $a \wedge b \in F$ .

We equip the set of filters with the *inclusion order*, i.e.,  $F \leq F'$  means  $F \subseteq F'$ .

**Definition 46.** A filter  $F$  in  $A$  is an *ultrafilter* if, for any  $a \in A$ , exactly one of  $a \in F$  or  $\neg a \in F$ .

**Remark 47.** A filter  $F$  is *proper* if  $\perp$  is not an element of  $F$ . An equivalent definition of ultrafilter is:  $F$  is a proper filter and, for any  $a \in A$ ,  $a \in F$  or  $\neg a \in F$ . Indeed, ultrafilters as defined in Definition 46 must be proper since  $\top \in F$ , so  $\perp = \neg\top \notin F$ . On the other hand, if  $F$  is proper, then we cannot have both  $a \in F$  and  $\neg a \in F$ , since then  $\perp = a \wedge \neg a \in F$ .

**Remark 48.** A filter  $F$  is *principal* if it has a minimum element, i.e.,  $F = \uparrow a$  for some  $a \in F$ . When  $A$  is a *finite* Boolean algebra, any ultrafilter is principal. More precisely, the ultrafilters of a finite Boolean algebra are exactly the subsets of the form  $\uparrow x$ , where  $x$  is an *atom*, i.e., a minimal element of  $A \setminus \{\perp\}$  ([exercise](#)). Consider the (infinite) Boolean algebra of finite-or-cofinite subsets of  $\mathbb{N}$ . The set of cofinite subsets is a non-principal ultrafilter in this algebra. In fact, infinite Boolean algebras always contain non-principal ultrafilters ([exercise](#)).

**Lemma 49** (Ultrafilter lemma). *Let  $F$  be a filter in a Boolean algebra  $A$  and let  $a \in A \setminus F$ . There exists an ultrafilter  $G$  in  $A$  such that  $F \subseteq G$  and  $a \notin G$ .*

*Proof.* Consider the poset  $P := \{F' \text{ a filter in } A \mid a \notin F'\}$ . This poset is non-empty since it contains  $F$ , and it is closed under unions of chains. By Zorn's Lemma, pick a maximal  $G \in P$ . We claim that  $G$  is an ultrafilter. First note that  $G$  is proper: if  $\perp \in G$  then also  $a \in G$  since  $\perp \leq a$ . Now let  $b \in A$  and suppose, towards a contradiction, that  $b \notin G$  and  $\neg b \notin G$ . Consider the filter  $G_1$  generated by  $G \cup \{b\}$  and the filter  $G_2$  generated by  $G \cup \{\neg b\}$ ; concretely,

$$\begin{aligned} G_1 &= \{c \in A \mid \text{there exists } g \in G \text{ such that } g \wedge b \leq c\} , \\ G_2 &= \{c \in A \mid \text{there exists } g \in G \text{ such that } g \wedge \neg b \leq c\} . \end{aligned}$$

For  $i = 1, 2$ , the filter  $G_i$  contains  $G$  strictly, so by maximality of  $G$ , we must have  $a \in G_i$ ; pick  $g_i \in G_i$  such that  $g_1 \wedge b \leq a$  and  $g_2 \wedge \neg b \leq a$ . Define  $g := g_1 \wedge g_2$ . Then  $g \in G$  since  $G$  is a filter. Now

$$g = (g \wedge b) \vee (g \wedge \neg b) \leq (g_1 \wedge b) \vee (g_2 \wedge \neg b) \leq a ,$$

so  $a \in G$ , contrary to the fact that  $G \in P$ . □

**Definition 50.** Let  $A$  be a Boolean algebra. Write  $W_A$  for the set of ultrafilters in  $A$ . We define the function  $\eta: A \rightarrow \mathcal{P}(W_A)$ , by, for  $a \in A$ ,

$$\eta(a) := \{w \in W_A \mid a \in w\} .$$

**Proposition 51** (Stone representation theorem for Boolean algebras). *For any Boolean algebra  $A$ , the function  $\eta: A \rightarrow \mathcal{P}(W_A)$  is an injective Boolean algebra homomorphism.*

*Proof.* Proving that  $\eta$  preserves  $\neg, \vee$  and  $\top$  is a useful [exercise](#). We focus on injectivity. For this, it suffices to show the special case that, if  $a \in A$  and  $a \neq \perp$ , then  $\eta(a) \neq \emptyset$ . Indeed, given this special case, if  $b, c \in A$  are arbitrary with  $b \neq c$ , then  $a := (\neg b \wedge c) \vee (b \wedge \neg c)$  is not  $\perp$ , and its image under  $\eta$  is  $(\eta(b)^c \cap \eta(c)) \cup (\eta(b) \cap \eta(c)^c)$ , which is then non-empty, so that  $\eta(b) \neq \eta(c)$ .

Let  $a \in A \setminus \{\perp\}$ . Consider  $F := \uparrow a$ . Then  $\neg a \notin F$ : if  $a \leq \neg a$  then  $\perp = a \wedge \neg a = a$ . By [Lemma 49](#), pick  $w \in W_A$  such that  $F \subseteq w$  and  $\neg a \notin w$ . Since  $w$  is an ultrafilter,  $a \in w$ . □

**Definition 52.** Let  $(A, \Box)$  be a modal algebra and  $W_A$  the set of characters of  $A$ . The *Jónsson-Tarski relation* is the binary relation  $R_A \subseteq W_A \times W_A$  defined, for  $w, v \in W_A$ , by

$$wR_A v \text{ if, and only if, for all } a \in A, \text{ if } w \in \eta(\Box a) \text{ then } v \in \eta(a).$$

The pair  $(W_A, R_A)$  is the *ultrafilter frame* of the modal algebra  $(A, \Box)$ .

**Remark 53.** For any filter  $F$  in  $A$ , write  $\Box^{-1}F := \{b \in A \mid \Box b \in F\}$ . Then  $\Box^{-1}(F)$  is also a filter in  $A$ . A different way of phrasing the definition of  $R_A$  is:  $wR_A v$  if, and only if,  $\Box^{-1}w \subseteq v$ .

**Proposition 54.** *The frame  $(W_A, R_A)$  of characters equipped with the Jónsson-Tarski relation and the Boolean algebra homomorphism  $\eta: A \rightarrow \mathcal{P}(W_A)$  satisfy (1).*

*Proof.* Let  $a \in A$  be arbitrary. We need to show that, for any  $w \in W$ ,

$$\Box a \in w \iff \text{for all } v \in W, \text{ if } wR_A v \text{ then } a \in v. \quad (3)$$

( $\Rightarrow$ ). Suppose  $\Box a \in w$  and  $v \in R_A[w]$ . Then  $w \in \eta(\Box a)$  so, by definition of  $R_A$ ,  $v \in \eta(a)$ , which means that  $a \in v$ .

( $\Leftarrow$ ). We prove the contrapositive. Suppose that  $\Box a \notin w$ . Then  $a$  is not an element of the filter  $\Box^{-1}w$ . By Lemma 49, pick  $v \in W_A$  such that  $\Box^{-1}w \subseteq v$  and  $a \notin v$ . By Remark 53,  $wR_A v$ . Thus,  $v$  witnesses that  $w$  is not in the right-hand-side of (1).  $\square$

**Remark 55.** The above proof method for completeness applies in many situations beyond basic modal logic  $\mathbf{K}$ . However, it is in general non-constructive, due to its use of the ultrafilter lemma (Lemma 49). In many cases (including  $\mathbf{K}$ ), constructive arguments do exist; for instance, one can often restrict to an appropriately chosen finite part  $\mathcal{L}'$  of the Lindenbaum algebra  $\mathcal{L}$  and then replace the ultrafilter frame by a frame based on the set of atoms of  $\mathcal{L}'$ . See e.g. [1, Section 2.3].

## References

- [1] P. Blackburn, M. de Rijke, and Y. Venema. *Modal Logic*, volume 53 of *Cambridge Tracts in Theoretical Computer Science*. Cambridge University Press, 2001.
- [2] M. Gehrke and S. van Gool. *Topological Duality for Distributive Lattices: Theory and Applications*, volume 61 of *Cambridge Tracts in Theoretical Computer Science*. Cambridge University Press, 2024.