LOGIQUE

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Informations pratiques

https://samvangool.net/logique.html

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Logic studies methods of reasoning.

In propositional logic, we have:

- atomic propositions $P, Q, R, \ldots$
- connectives $\wedge, \rightarrow, \vee, \neg, \perp, \ldots$

In First-order (predicate) logic, we have the above, and:

- quantifiers $\forall, \exists$
- variables $x, y, \ldots$
- function symbols $f, g, \ldots$
- predicate symbols $R, S, \leq, \ldots$
- equality $=$

- truth tables
- Kripke models

- (Tarski) structures
- models of a first-order theory

An important theme in logic: a tension between syntax and semantics.

Syntactic methods are grouped as proof theory, semantic ones as model theory.

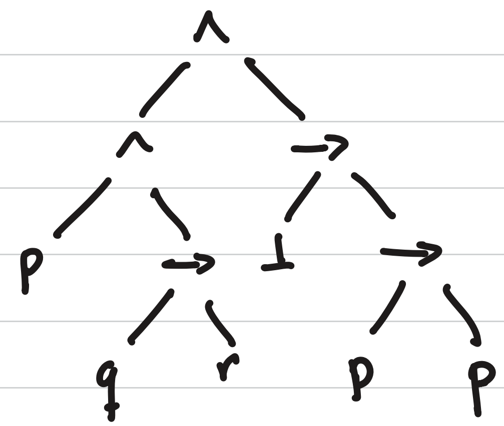
Many interesting results in logic happen at the interface of syntax & semantics.

1. Syntax

How to build propositional and first-order formulas

Let P_0 be a set, whose elements we will call atomic propositions.

- A (propositional) formula with atomic propositions in P_0 is an expression built inductively by applying:
    - for any $p \in P$, p is a formula;
    - for any formulas φ and ψ, $\varphi \wedge \psi$ is a formula;
    - for any formulas φ and ψ, $\varphi \to \psi$ is a formula;
    - $\perp$ is a formula.
- We write $\mathrm{Form}(P_0)$ for the set of formulas with atomic propositions in P_0.
- For example, if $P_0 = \{p, q, r\}$, then $\varphi := (p \wedge (q \to r)) \wedge (\perp \to (p \to p))$ is a formula.
- Any formula has a syntax tree, for instance, the syntax tree of φ:



- We define some abbreviations, for any formulas φ, ψ:

$$\neg\varphi := \varphi \to \perp \qquad \top := \perp \to \perp$$

$$\varphi \vee \psi := \neg(\neg\varphi \wedge \neg\psi) \qquad \varphi \leftrightarrow \psi := (\varphi \to \psi) \wedge (\psi \to \varphi)$$

$$\varphi \oplus \psi := \varphi \text{ XOR } \psi := (\varphi \wedge \neg\psi) \vee (\neg\varphi \wedge \psi)$$

- The above definition of propositional formula is an example of an inductive (or recursive) definition: the concept is defined in terms of itself.
  This is legal, provided that the "building blocks" are "smaller" than the thing being defined.
  We take the idea of inductive definitions as basic, but fundamental. We obtain from it a useful:

Induction Principle Let $X \subseteq \mathrm{Form}(P_0)$, and suppose that:

- for every $p \in P_0$, $p \in X$;
- for every $\varphi, \psi \in X$, $\varphi \wedge \psi \in X$ and $\varphi \to \psi \in X$; and
- $\bot \in X$.

Then $X = \mathrm{Form}(P_0)$.

- There is no priority between connectives. The expression $p \wedge q \to r$ is ambiguous, as it can be read as either $(p \wedge q) \to r$ or $p \wedge (q \to r)$. Some authors define conventions to reduce parentheses, but we mostly prefer to write them. Exception: $\neg p \wedge q$ means $(\neg p) \wedge q$, not $\neg(p \wedge q)$.

- A **signature** is a tuple $L = (F, P, ar)$ where F and P are (disjoint) sets and $ar : F \cup P \to \mathbb{N}$.
- We call elements of F **function symbols**, elements of P **predicate symbols** or **relation symbols**, and for $s \in F \cup P$, we call $ar(s)$ the **arity** of s.
- For $n \in \mathbb{N}$, we write $F_n := F \cap ar^{-1}(n)$ and $P_n := P \cap ar^{-1}(n)$.
- Let V be a set, whose elements we call **variables**.

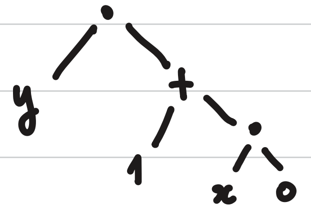
Term(F,V)

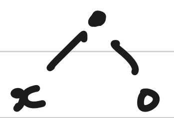
A **term** with functions in F and variables in V is defined inductively by:

- for any $x \in V$, x is a term
- for any $f \in F_n$ and any n-tuple of terms $(t_1, \dots, t_n)$, $f(t_1, \dots, t_n)$ is a term.

Example. Let $F_2 := \{+, \cdot\}$, $F_0 := \{0, 1\}$ and $F := F_0 \cup F_2$. Let $V := \{x, y\}$.

Then $\cdot(y, +(1, \cdot(x, 0)))$ is a term. We also write this as $y \cdot (1 + (x \cdot 0))$ (infix notation).

We can draw **syntax trees** for terms. An **occurrence** of a term s in a term t is a node in the syntax tree of t that generates the syntax tree of s. E.g. 

has an occurrence of 

.

- A (first-order) formula in signature $L = (F, P, ar)$ and variables in V is defined inductively by:
    - for any $R \in P_n$ and any n-tuple of (F,V)-terms $t_1, \dots, t_n$, $R(t_1, \dots, t_n)$ is an (atomic) formula,
    - (*) for any pair of terms s, t, $s = t$ is an (atomic) formula,
    - for any formulas φ, ψ, $\varphi \wedge \psi$ and $\varphi \to \psi$ are formulas,
    - $\perp$ is a formula,
    - for any variable $x \in V$ and formula φ, $\forall x\, \varphi$ is a formula.

      (NB: this is not an "occurrence" of x)

We write $\mathrm{Form}(L, V)$ for the set of such formulas. (NB: same notation as propositional formulas!)

- The same abbreviations for $\neg, \vee, \top, \leftrightarrow$, XOR apply here, and:

$$\exists x\, \varphi := \neg(\forall x(\neg \varphi)).$$

- Some authors omit (*). We call the resulting concept a (first-order) formula without equality.
- Let $\varphi \in \mathrm{Form}(L, V)$ and $x \in V$. An occurrence of x in φ is bound if it is in the scope of some $\forall x$, and free otherwise. (NB: x may have free and bound occurrences in φ, or not occur in φ at all!)

$ar(\leq) := 2$

Example. Take F as before and $P := \{\leq\}$. Then $(\forall x(\leq(x, \cdot(y,1)))) \wedge (x=y)$ is a formula, written in a more readable way: $(\forall x\,(x \leq y \cdot 1)) \wedge (x=y)$. y has two free occurrences, x has a bound and a free occurrence.

Zero-o-logical remarks

- If $F = \emptyset$, then the only terms are variables: "purely relational" signature.
- If $P = \emptyset$ then the only atomic formulas are equalities: "purely functional" or "algebraic" signature.
- Elements of F_0 are called constant symbols.
- Elements of P_0 are called atomic propositions.
- If $F = \emptyset$ and $P_n = \emptyset$ for all $n > 0$, then taking $V = \emptyset$ we see that first-order formulas in signature L and without variables are the same thing as propositional formulas with atomics in P_0.
- A formula with no free occurrences of variables is called a sentence or closed formula.

Exercise. A propositional formula with atomics in P_0 is also the same thing as a term with functions in $F := F_0 \cup F_2$ where $F_0 := \{\bot\}$ and $F_2 := \{\wedge, \rightarrow\}$, and variables in P_0.

2. Semantics

What it all means.

Let P_0 be a set of atomic propositions. We write $2 := \{0,1\}$.

An interpretation is a function $M : P_0 \to 2$.

Given an interpretation M, we inductively define, for every $\varphi \in \mathrm{Form}(P_0)$, a truth value, $[\![\varphi]\!]_M$, by:

- for $p \in P_0$, $[\![p]\!]_M := M(p)$;
- for $\varphi, \psi \in \mathrm{Form}(P_0)$, $[\![\varphi \wedge \psi]\!]_M := [\![\varphi]\!]_M \cdot [\![\psi]\!]_M$;
- —— " ——, $[\![\varphi \to \psi]\!]_M := \max(1 - [\![\varphi]\!]_M \cdot [\![\psi]\!]_M) = \begin{cases} 0 & \text{if } [\![\varphi]\!]_M = 1 \text{ and } [\![\psi]\!]_M = 0, \\ 1 & \text{otherwise.} \end{cases}$
- $[\![\bot]\!]_M := 0$.

Exercise. For any $\varphi, \psi \in \mathrm{Form}(P_0)$, we have:

$[\![\varphi \vee \psi]\!]_M = \max([\![\varphi]\!]_M, [\![\psi]\!]_M)$, $[\![\neg\varphi]\!]_M = 1 - [\![\varphi]\!]_M$, $[\![\top]\!]_M = 1$,

$[\![\varphi \text{ XOR } \psi]\!]_M = [\![\varphi]\!]_M + [\![\psi]\!]_M \bmod 2$, $[\![\varphi \leftrightarrow \psi]\!]_M = 1 - ([\![\varphi]\!]_M - [\![\psi]\!]_M)^2 = \begin{cases} 1 & \text{if } [\![\varphi]\!]_M = [\![\psi]\!]_M, \\ 0 & \text{otherwise.} \end{cases}$

We write $M \models \varphi$ if $[\![\varphi]\!]_M = 1$ and $M \not\models \varphi$ if $[\![\varphi]\!]_M = 0$.

In short: $[\![\varphi]\!]_M$ is the evaluation of φ in the Boolean algebra 2, substituting $M(p)$ for each occurrence of p.

We call a propositional formula φ **valid** if, for all interpretations M, $M \models \varphi$,

satisfiable if there exists an interpretation M such that $M \models \varphi$,

unsatisfiable if, for all interpretations M, $M \not\models \varphi$.

Observation. φ is unsatisfiable if, and only if, $\neg\varphi$ is valid.

Examples.

$$\varphi_1 := (p \to q) \vee (q \to p)$$

$$\varphi_2 := \neg\neg p \to p$$

$$\varphi_3 := p \wedge ((q \vee r) \wedge (\neg q \vee r) \wedge (\neg r \vee s)) \wedge \neg p$$

Valid? Unsatisfiable?

Proposition	RW's Estimated Likelihood
TRUE	100%
$\mathsf{EXP}^{\mathsf{NP}} \neq \mathsf{BPP}$	99%
$\mathsf{NEXP} \not\subset \mathsf{P/poly}$	97%
$\mathsf{L} \neq \mathsf{NP}$	95%
$\mathsf{NP} \not\subset \mathsf{SIZE}(n^k)$	93%
$\mathsf{BPP} \subseteq \mathsf{SUBEXP}$	90%
$\mathsf{P} \neq \mathsf{PSPACE}$	90%
$\mathsf{P} \neq \mathsf{NP}$	80%
ETH	70%
$\mathsf{NC1} \neq \mathsf{TC0}$	50%
$\mathsf{NEXP} \neq \mathsf{EXP}$	45%
SETH	25%
$\mathsf{NEXP} \neq \mathsf{coNEXP}$	20%
NSETH	15%
$\mathsf{L} \neq \mathsf{RL}$	5%
FALSE	0%

Table 1. What you receive, when you ask for my opinions on some open problems in complexity theory.

Theorem (Cook & Levin) The decision problem "$\overbrace{\text{given } \varphi, \text{ is } \varphi \text{ satisfiable?}}^{\text{SAT}}$" is NP-complete, even for $\#P_0 = 3$.

In practice, there exist good algorithms, both theoretical and practical ("SAT-solvers").

In theory, there might exist a deterministic polynomial-time algorithm for SAT (although most experts don't think so).

The question is called **P vs NP**. A problem even money cannot solve (so far).

R. R. Williams, "Some Estimated Likelihoods for Computational Complexity", LNCS 10000 (2018) https://people.csail.mit.edu/rrw/likelihoods.pdf

We now extend the semantics from propositional to first-order logic.

Let $L = (F, P, ar)$ be a signature. An L-structure is a tuple $\mathcal{M} = (M, (f^{\mathcal{M}})_{f \in F}, (R^{\mathcal{M}})_{R \in P})$

where:
- M is a set,
- for each $f \in F_n$, $f^{\mathcal{M}}$ is a function $M^n \longrightarrow M$,
- for each $R \in P_n$, $R^{\mathcal{M}}$ is a subset of M^n.

Q. Why does it "extend" the propositional case?

A. When $F = \emptyset$ and $P_n = 0$ for all $n > 0$, $\mathcal{M} = (M, (p^{\mathcal{M}})_{p \in P_0})$, where, for each $p \in P_0$, $p^{\mathcal{M}} \subseteq M^0$. Since $\# M_0 = 1$, $\# \mathcal{P}(M_0) = 2$. So we can see $\mathcal{M}$ as a function $P_0 \longrightarrow 2$.

Let V be a set of variables. A valuation of V in $\mathcal{M}$ is a function $v : V \rightarrow M$.

A valuation v of V in $\mathcal{M}$ extends (uniquely) to a homomorphism $\bar{v} : \mathrm{Term}(F, V) \longrightarrow M$, by induction:
- for $x \in V$, $\bar{v}(x) := v(x)$, and
- for $f \in F_n$ and $(t_1, \ldots, t_n) \in \mathrm{Term}(F, V)^n$, $\bar{v}(f(t_1, \ldots, t_n)) := f^{\mathcal{M}}(\bar{v}(t_1), \ldots, \bar{v}(t_n))$.

(Check that the second rule is well-typed!)

Let $\mathcal{M}$ be an L-structure, V a set of variables, $v : V \longrightarrow M$ a valuation of V in $\mathcal{M}$.

When $x \in V$ and $m \in \mathcal{M}$, we define $v[x \mapsto m] := \lambda y.$ if $x = y$ then m else $v(y)$.

In words, $v[x \mapsto m]$ is the valuation v in which we update the value at x to be m.

For a formula $\varphi \in \mathrm{Form}(L, V)$, its truth value, $[\![\varphi]\!]_{\mathcal{M},v}$ under the valuation v is defined inductively:

- for $R \in P_n$ and $(t_1, \dots, t_n) \in \mathrm{Term}(F,V)^n$, $[\![R(t_1, \dots, t_n)]\!]_{\mathcal{M},v} := 1$ if $(\bar{v}(t_1), \dots, \bar{v}(t_n)) \in R^{\mathcal{M}}$, 0 otherwise;
- for $s, t \in \mathrm{Term}(F,V)$, $[\![s = t]\!]_{\mathcal{M},v} := 1$ if $\bar{v}(s) = \bar{v}(t)$, 0 otherwise;
- for $\varphi, \psi \in \mathrm{Form}(L,V)$, $[\![\varphi \wedge \psi]\!]_{\mathcal{M},v} := [\![\varphi]\!]_{\mathcal{M},v} \cdot [\![\psi]\!]_{\mathcal{M},v}$ and $[\![\varphi \to \psi]\!]_{\mathcal{M},v} := \max(1 - [\![\varphi]\!]_{\mathcal{M},v}, [\![\psi]\!]_{\mathcal{M},v})$;
- for $\varphi \in \mathrm{Form}(L,V)$ and $x \in V$, $[\![\forall x\, \varphi]\!]_{\mathcal{M},v} := 1$ if for every $m \in M$, $[\![\varphi]\!]_{\mathcal{M},v[x \mapsto m]} = 1$, 0 otherwise.

We write $\mathcal{M}, v \models \varphi$ or $\mathcal{M} \models_v \varphi$ if $[\![\varphi]\!]_{\mathcal{M},v} = 1$, and $\mathcal{M}, v \not\models \varphi$ or $\mathcal{M} \not\models_v \varphi$ if $[\![\varphi]\!]_{\mathcal{M},v} = 0$.

Exercise. $[\![\exists x\, \varphi]\!]_{\mathcal{M},v} = 1$ iff there exists $m \in M$ such that $[\![\varphi]\!]_{\mathcal{M},v[x \mapsto m]} = 1$.

(and of course the analogous facts hold for $\neg, \vee, \top, \longleftrightarrow$, XOR, as in the propositional setting.)

Lemma. If a variable x has no free occurrence in φ, then $[\![\varphi]\!]_{\mathcal{M},v}$ does not depend on the value of v at x,

i.e., if v and v' are valuations with $v(y) = v'(y)$ for all $y \in V \setminus \{x\}$, then $[\![\varphi]\!]_{\mathcal{M},v} = [\![\varphi]\!]_{\mathcal{M},v'}$.

Proof. 1) We first show by induction that, for any term in which t does not occur, $\bar{v}(t) = \bar{v}'(t)$

2) We next show by induction on φ that $[\![\varphi]\!]_{\mathcal{M},v} = [\![\varphi]\!]_{\mathcal{M},v'}$.

This is a good exercise in using inductive definitions! $\square$

Because of the above Lemma, when evaluating $[\![\varphi]\!]_{\mathcal{M},v}$, we may restrict v to the variables occurring freely in φ.

Notation. "Let $\varphi(x_1, \dots, x_n)$ be a formula" means: "Let φ be a formula such that, if x occurs freely in φ, then $x \in \{x_1, \dots, x_n\}$."

Let $\varphi(x_1, \dots, x_n)$ be a formula, and $\bar{m} := (m_1, \dots, m_n) \in M^n$.

We write $\mathcal{M}, \bar{m} \models \varphi$ if $\mathcal{M}, v \models \varphi$, where v is any valuation such that

$v(x_i) = m_i$ for every $1 \leq i \leq n$.

When φ is a sentence, we write $\mathcal{M} \models \varphi$ (since $\bar{m}$ is then an empty tuple).

Some authors extend this notation: $\mathcal{M} \models \varphi$ means $\mathcal{M} \models_v \varphi$ for all valuations φ.

Example. As before, let $L = (F, P, ar)$ with $F = \{+, \cdot, 0, 1\}$ and $P = \{\leq\}$.

Consider $\mathcal{N} = (\mathbb{N}, +^{\mathcal{N}}, \cdot^{\mathcal{N}}, 0^{\mathcal{N}}, 1^{\mathcal{N}}, \leq^{\mathcal{N}})$ where, for any $m, n \in \mathbb{N}$,

$$\begin{aligned} m +^{\mathcal{N}} n &:= m + n \\ m \cdot^{\mathcal{N}} n &:= m \cdot n \\ 0^{\mathcal{N}} &:= 0 \\ 1^{\mathcal{N}} &:= 1 \\ \leq^{\mathcal{N}} &:= \{(m,n) \in \mathbb{N}^2 \mid m \leq n\}. \end{aligned}$$

Then $\mathcal{N}$ is an L-structure, satisfying familiar facts such as:

$$\mathcal{N} \models \forall x \forall y (x + y = y + x) \text{ and } \mathcal{N} \models \forall x (x \cdot 0 = 0) \text{ and } \mathcal{N} \models \forall x (0 \leq x)$$

However, nothing so far prevents us from considering the L-structure

$\mathcal{M} := (\mathbb{N}, +^{\mathcal{M}}, \cdot^{\mathcal{M}}, 0^{\mathcal{M}}, 1^{\mathcal{M}}, \leq^{\mathcal{M}})$ where, for any $m, n \in \mathbb{N}$,

$$\begin{aligned} m +^{\mathcal{M}} n &:= m^2 \\ m \cdot^{\mathcal{M}} n &:= 42 \\ 0^{\mathcal{M}} &:= 2026 \\ 1^{\mathcal{M}} &:= 1 \\ \leq^{\mathcal{M}} &:= \{(m,n) \in \mathbb{N}^2 / m < n\} \end{aligned}$$

In this (unusual) L-structure, none of the above sentences are valid, while, for example $\mathcal{M} \models 1 \leq 0$, and $\mathcal{M} \models \forall x (\neg (x \cdot 0 = 0))$.

Still, certain sentences are true in all structures, e.g., $\forall x (x \leq y \vee \neg (x \leq y))$.

Such sentences are called tautologies.

In practice, we are often interested not in "pure" tautologies, but in consequences of a theory.

A theory in a signature L is a set of L-sentences.

A model of a theory T is an L-structure $\mathcal{M}$ such that, for every $\varphi \in T$, $\mathcal{M} \models \varphi$.

A (semantic) consequence of a theory T is a sentence φ such that, for any model $\mathcal{M}$ of T, it is the case that $\mathcal{M} \models \varphi$. Notation: '$T \models \varphi$'. When $T = \{\psi\}$: '$\psi \models \varphi$'. When $T = \emptyset$: '$\models \varphi$'.

Sentences φ and ψ are (semantically) equivalent if both $\varphi \models \psi$ and $\psi \models \varphi$. Notation: '$\varphi \equiv \psi$'.

Example. Let $L = (\{\cdot, 1\}, \emptyset, \mathrm{ar}: (\cdot \mapsto 2, 1 \mapsto 0))$, and $T := \{\forall x\,(x \cdot 1 = x), \forall x\,(1 \cdot x = x), \forall x \forall y \forall z\,((x \cdot y) \cdot z = x \cdot (y \cdot z))\}$.

Then a model of T is the same thing as a monoid.

$T \models \varphi$ means: φ is true in all monoids.

For example, $T \models 1 \cdot 1 = 1$, and $T \models \forall x\,((x \cdot x) \cdot x = x \cdot (x \cdot x))$.

However, $T \not\models \forall x \forall y\,(x \cdot y = y \cdot x)$, because there exist non-commutative monoids.

(In this example, T is finite, but there exist many infinite theories of interest. For instance, the theories PA (Peano Arithmetic) or ZF (Zermelo-Fraenkel) are infinite theories in the sense given here.)

German for "decision problem".

Entscheidungsproblem. Given a sentence φ as input, decide whether or not $\models \varphi$.

(Hilbert & Ackermann, 1928)

This is not just any decision problem. It's the first one ever shown to be undecidable:

Theorem There is no algorithm for solving the Entscheidungsproblem.

(Church & Turing, independently, both in 1936)

Proof (Sketch) One can define a signature L, and, for every $n \in \mathbb{N}$, an L-sentence φ_n, such that

$\models \varphi_n$ if, and only if, the n^{th} Turing machine halts on all inputs

(or: the n^{th} computable function is total). $\square$

David Hilbert (1862-1943)

Alan Turing (1912-1954)

Alonzo Church (1903-1995)

3. Natural deduction

How to prove things, formally.

We now return to the syntactic side, and define what it means to prove something.

In fact, there are many possible definitions of "proof", and we call a proof system or calculus one such choice.

We begin with a system called natural deduction, and we first look at propositional logic.

Let P_0 be a set of atomic propositions. A sequent is an element of $\mathcal{P}(\mathrm{Form}(P_0)) \times \mathrm{Form}(P_0)$.

We will denote such a pair by $\Gamma \Rightarrow \varphi$. (Capital Greek letter for "set of formulas", lowercase Greek letter for "formula". The arrow $\Rightarrow$ is just a notation.)

We define the set $\mathcal{D}$ of (natural deduction) derivable sequents inductively by:

- (Ax) if $\varphi \in \Gamma$, then $\Gamma \Rightarrow \varphi$ is derivable;
- ($\wedge$I) if $\Gamma_1 \Rightarrow \varphi_1$ and $\Gamma_2 \Rightarrow \varphi_2$ are derivable, then $\Gamma_1 \cup \Gamma_2 \Rightarrow \varphi_1 \wedge \varphi_2$ is derivable;
- ($\wedge$E) if $\Gamma \Rightarrow \varphi \wedge \psi$ is derivable, then $\Gamma \Rightarrow \varphi$ is derivable and $\Gamma \Rightarrow \psi$ is derivable;
- ($\rightarrow$I) if $\Gamma \cup \{\varphi\} \Rightarrow \psi$ is derivable, then $\Gamma \Rightarrow \varphi \rightarrow \psi$ is derivable,
- ($\rightarrow$E) if $\Gamma_1 \Rightarrow \varphi$ is derivable and $\Gamma_2 \Rightarrow \varphi \rightarrow \psi$ is derivable, then $\Gamma_1 \cup \Gamma_2 \Rightarrow \psi$ is derivable.
- ($\bot$E) if $\Gamma \Rightarrow \bot$ is derivable, then, for any φ, $\Gamma \Rightarrow \varphi$ is derivable.
- (C) if $\Gamma \cup \{\neg\varphi\} \Rightarrow \bot$ is derivable, then $\Gamma \Rightarrow \varphi$ is derivable.

Example. $\{p\} \Rightarrow q \to p$ is derivable, for any $p, q \in P_0$.

Proof. By (Ax), $\{p, q\} \Rightarrow p$ is derivable. By $(\to I)$, $\{p\} \Rightarrow q \to p$ is derivable. □

We define the notation $\Gamma \vdash \varphi$ for: "the sequent $\Gamma \Rightarrow \varphi$ is derivable".

We can now write the definition of $\vdash$ more succinctly as:

$$(Ax)\ \frac{\varphi \in \Gamma}{\Gamma \vdash \varphi} \qquad (\wedge I)\ \frac{\Gamma_1 \vdash \varphi \qquad \Gamma_2 \vdash \psi}{\Gamma_1, \Gamma_2 \vdash \varphi \wedge \psi} \qquad (\wedge E_L)\ \frac{\Gamma \vdash \varphi \wedge \psi}{\Gamma \vdash \varphi} \qquad (\wedge E_R)\ \frac{\Gamma \vdash \varphi \wedge \psi}{\Gamma \vdash \psi} \qquad (\bot E)\ \frac{\Gamma \vdash \bot}{\Gamma \vdash \varphi}$$

$$(\to I)\ \frac{\Gamma, \varphi \vdash \psi}{\Gamma \vdash \varphi \to \psi} \qquad (\to E)\ \frac{\Gamma_1 \vdash \varphi \qquad \Gamma_2 \vdash \varphi \to \psi}{\Gamma_1, \Gamma_2 \vdash \psi} \qquad (C)\ \frac{\Gamma, \neg\varphi \vdash \bot}{\Gamma \vdash \varphi}$$

→ "proof by contradiction"

Example.

$$(\to I)\ \dfrac{(Ax)\ \dfrac{p \in \{p, q\}}{p, q \vdash p}}{p \vdash q \to p}$$

← usually omitted

Notation: $\dfrac{A}{B}$ means: "if A, then B".

"premises" → $\dfrac{A_1 \quad A_2}{B}$ ← "conclusion" means "if A_1 and A_2, then B"

Γ, φ means "$\Gamma \cup \{\varphi\}$" and Γ_1, Γ_2 means "$\Gamma_1 \cup \Gamma_2$"

Example. $\vdash \varphi \to (\neg\varphi \to \psi)$ for any formulas φ, ψ.

$\Gamma = \emptyset$

Proof.

$$
(\to I)\,\dfrac{(\to I)\,\dfrac{(\bot E)\,\dfrac{(\to E)\,\dfrac{(Ax)\,\dfrac{}{\varphi \vdash \varphi} \qquad (Ax)\,\dfrac{}{\neg\varphi \vdash \neg\varphi}}{\varphi, \neg\varphi \vdash \bot}}{\varphi, \neg\varphi \vdash \psi}}{\varphi \vdash \neg\varphi \to \psi}}{\vdash \varphi \to (\neg\varphi \to \psi)}
$$

$$
\dfrac{\overset{\Gamma_1}{\varphi} \vdash \varphi \qquad \overset{\Gamma_2}{\varphi \to \bot} \vdash \varphi \to \overset{\psi}{\bot}}{\underset{\Gamma_1, \Gamma_2}{\varphi, \varphi \to \bot} \vdash \underset{\psi}{\bot}}
$$

Example $\neg\neg\varphi \vdash \varphi$ for any formula φ

$$
(C)\,\dfrac{(\to E)\,\dfrac{(Ax)\,\dfrac{}{\neg\neg\varphi \vdash \neg\neg\varphi} \qquad (Ax)\,\dfrac{}{\neg\varphi \vdash \neg\varphi}}{\neg\neg\varphi, \neg\varphi \vdash \bot}}{\neg\neg\varphi \vdash \varphi}
$$

$$
\dfrac{\Gamma_1 \vdash \varphi' \to \psi \qquad \Gamma_2 \vdash \varphi'}{\Gamma_1 \cup \Gamma_2 \vdash \bot}
$$

$\Gamma_1 = \{\neg\neg\varphi\}$
$\Gamma_2 = \{\neg\varphi\}$
$\varphi' = \varphi \to \bot$
$\psi = \bot$

There exist other notations for Natural Deduction proofs, due to Jaśkowski, Fitch, Suppes.

Example. $\varphi \vdash \neg\varphi \to \psi$. The proof in Suppes-style:

$\{1\}$	1.	φ	(Ax)
$\{2\}$	2.	$\neg\varphi$	(Ax)
$\{1,2\}$	3.	$\bot$	($\to$E 1,2)
$\{1,2\}$	4.	ψ	($\bot$E 3)
$\{1\}$	5.	$\neg\varphi \to \psi$	($\to$I 2,4)

Each line i has form "S i. θ_i (R)", where S is a set of indices $\leq i$, and R is a rule together with indices pointing to where the premises are found.

To convert this notation into a derivation like before, read each line as a sequent $\{\theta_j : j \in S\} \Rightarrow \theta_i$:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{1.\ \varphi \vdash \varphi}\,(Ax) \qquad \dfrac{}{2.\ \neg\varphi \vdash \neg\varphi}\,(Ax)}{3.\ \varphi, \neg\varphi \vdash \bot}}{4.\ \varphi, \neg\varphi \vdash \psi}\,(\bot E)}{5.\ \varphi \vdash \neg\varphi \to \psi}\,(\to I)$$

Fitch-style:

$$\begin{array}{|l} \varphi \\ \begin{array}{|l} \neg\varphi \\ \bot \\ \psi \end{array} \\ \neg\varphi \to \psi \end{array}$$