LOGIQUE

ENS Paris-Saclay

DER d'informatique

cours 3, 2025-26

Point planning :

- pas de cours le 4 mars
- 11 mars : double cours $14^h - 16^h$ et $16^h15 - 18^h15$
- à partir du 18 mars : 14-16 TD

  $16^{15} - 18^{15}$ cours
- le DM sera publié le 25 mars, à rendre le 6 avril avant 23^h59.

6. First-order natural deduction

Adding rules for quantifiers and equality

Recall the syntax of FO logic (cours 1):

- A **signature** is a tuple $L = (F, P, ar)$ where F and P are (disjoint) sets and $ar : F \cup P \to \mathbb{N}$.
- We call elements of F **function symbols**, elements of P **predicate symbols** or **relation symbols**, and for $s \in F \cup P$, we call $ar(s)$ the **arity** of s.
- For $n \in \mathbb{N}$, we write $F_n := F \cap ar^{-1}(n)$ and $P_n := P \cap ar^{-1}(n)$.
- Let V be a set, whose elements we call **variables**.

Term(F,V)

A **term** with functions in F and variables in V is defined inductively by:

- for any $x \in V$, x is a term
- for any $f \in F_n$ and any n-tuple of terms $(t_1, \dots, t_n)$, $f(t_1, \dots, t_n)$ is a term.

- A first-order formula in signature $L = (F, P, ar)$ and variables in V is defined inductively by:
    - for any $R \in P_n$ and any n-tuple of (F,V)-terms $t_1, \dots, t_n$, $R(t_1, \dots, t_n)$ is an (atomic) formula,
    - (∗) for any pair of terms s, t, $s \doteq t$ is an (atomic) formula, (we often write $=$ for $\doteq$)
    - for any formulas φ, ψ, $\varphi \wedge \psi$ and $\varphi \to \psi$ are formulas,
    - $\bot$ is a formula,
    - for any variable $x \in V$ and formula φ, $\forall x\, \varphi$ is a formula.

  We write $\mathrm{Form}(L, V)$ for the set of such formulas. We will always assume V is infinite.

- The same abbreviations for $\neg, \vee, \top, \leftrightarrow, \mathrm{XOR}$ apply here, and:

$$\exists x\, \varphi \;:=\; \neg(\forall x(\neg \varphi)).$$

- Some authors omit (∗). We call the resulting concept a (first-order) formula without equality.
- Let $\varphi \in \mathrm{Form}(L, V)$ and $x \in V$. An occurrence of x in φ is bound if it is in the scope of some $\forall x$, and free otherwise. (NB: x may have free <u>and</u> bound occurrences in φ, or not occur in φ at all!)

Semantics (recall from cours 1)

Let $L = (F, P, ar)$ be a signature. An L-structure is a tuple $\mathcal{M} = (M, (f^M)_{f \in F}, (R^M)_{R \in P})$ where M is a non-empty set, for each $f \in F_n$, $f^{\mathcal{M}}$ is a function $M^n \longrightarrow M$, and for each $R \in P_n$, $R^{\mathcal{M}}$ is a subset of M^n.

Let V be a set of variables. A valuation of V in $\mathcal{M}$ is a function $v : V \to M$.

A valuation v of V in $\mathcal{M}$ extends (uniquely) to a homomorphism $\bar{v} : \mathrm{Term}(F, V) \longrightarrow M$, by induction:

- for $x \in V$, $\bar{v}(x) := v(x)$, and
- for $f \in F_n$ and $(t_1, \dots, t_n) \in \mathrm{Term}(F, V)^n$, $\bar{v}(f(t_1, \dots, t_n)) := f^{\mathcal{M}}(\bar{v}(t_1), \dots, \bar{v}(t_n))$.

We define $\mathcal{M}, v \vDash \varphi$ by induction on φ, with base cases:

$\mathcal{M}, v \vDash s \doteq t$ iff $\bar{v}(s) = \bar{v}(t)$ and $\mathcal{M}, v \vDash R(t_1, \dots, t_n)$ iff $R^{\mathcal{M}}$ contains $(\bar{v}(t_1), \dots, \bar{v}(t_n))$

and the inductive case for $\forall$:

$\mathcal{M}, v \vDash \forall x \varphi$ iff for every $a \in M$, $\mathcal{M}, v' \vDash \varphi$, where $v'(y) := \begin{cases} a & \text{if } y = x \\ v(y) & \text{otherwise} \end{cases}$.

We enrich the natural deduction system for propositional logic with two quantifier rules and two equality rules:

$$\frac{\Gamma \vdash \forall x\, \varphi}{\Gamma \vdash \varphi[t/x]}\ (\forall E)$$

and, if y does not occur freely in any $\gamma \in \Gamma \cup \{\forall x \varphi\}$:

$$\frac{\Gamma \vdash \varphi[y/x]}{\Gamma \vdash \forall x\, \varphi}\ (\forall I)$$

$$\frac{}{\Gamma \vdash t \doteq t}\ (\text{REFL}) \qquad \frac{\Gamma \vdash \varphi[t/x]}{\Gamma, t \doteq t' \vdash \varphi[t'/x]}\ (\text{SUBST})$$

- t, t' are arbitrary terms
- $\varphi[t/x]$ is the result of substituting t for x in φ (definition follows).

Definition of substitution of terms into terms.

Let $x_1, \dots, x_n$ be pairwise distinct variables and let $t_1, \dots, t_n$ be terms.

For any term t, we define $t[{}^{t_1}/x_1, \dots, {}^{t_n}/x_n]$ by induction on t:

- if t is a variable x,

$$t[{}^{t_1}/x_1, \dots, {}^{t_n}/x_n] := \begin{cases} t_i & \text{if } x = x_i \text{ for some } 1 \leq i \leq n \\ x & \text{otherwise} \end{cases}$$

- if $t = f(s_1, \dots, s_m)$ for f an n-ary function symbol:

$$t[{}^{t_1}/x_1, \dots, {}^{t_n}/x_n] := f\left(s_1[{}^{t_1}/x_1, \dots, {}^{t_n}/x_n], \dots, s_m[{}^{t_1}/x_1, \dots, {}^{t_n}/x_n]\right).$$

(In particular, if c is a constant, $c[{}^{t_1}/x_1, \dots, {}^{t_n}/x_n] = c$.)

Definition of substitution of terms into formulas.

Let $x_1, \dots, x_n$ be distinct variables and $t_1, \dots, t_n$ terms.

We define $\varphi[{}^{t_1}/x_1, \dots, {}^{t_n}/x_n]$ by induction on φ:

- if φ is $R(s_1, \dots, s_m)$ for an m-ary relation symbol R,
$$\varphi[{}^{t_1}/x_1, \dots, {}^{t_n}/x_n] := R(s_1[{}^{t_1}/x_1, \dots, {}^{t_n}/x_n], \dots, s_m[{}^{t_1}/x_1, \dots, {}^{t_n}/x_n])$$
- if φ is $s_1 \doteq s_2$, then $\varphi[{}^{t_1}/x_1, \dots {}^{t_n}/x_n] := s_1[{}^{t_1}/x_1, \dots, {}^{t_n}/x_n] \doteq s_2[{}^{t_1}/x_1, \dots, {}^{t_n}/x_n]$.
- if φ is $\psi_1 \to \psi_2$, then $\varphi[{}^{t_1}/x_1, \dots, {}^{t_n}/x_n] := \psi_1[{}^{t_1}/x_1, \dots, {}^{t_n}/x_n] \to \psi_2[{}^{t_1}/x_1, \dots, {}^{t_n}/x_n]$.
- if φ is $\perp$ then $\varphi[{}^{t_1}/x_1, \dots, {}^{t_n}/x_n] := \perp$
- if φ is $\forall x \psi$ then
$$\varphi[{}^{t_1}/x_1, \dots, {}^{t_n}/x_n] := \forall u\, \psi[{}^{t_{i_1}}/x_{i_1}, \dots, {}^{t_{i_k}}/x_{i_k}, {}^{u}/x],$$

where $x_{i_1}, \dots, x_{i_k}$ are those variables among $x_1, \dots, x_n$ such that x_i is free in $\forall x \psi$ and $x_i \neq t_i$,

and $u := \begin{cases} x, \text{ if } x \text{ does not occur in } t_{i_1}, \dots, t_{i_k}; \\ \text{a variable not occurring in } \psi, t_{i_1}, \dots, t_{i_k}, \text{ otherwise.} \end{cases}$

· In the last clause of the preceding definition, we need to be careful to avoid capture:

Example.

1) $(\forall x\, P(x,y))[x/y] \neq \forall x\, P(x,x)$ because the x in the substituting term would be captured

Instead: $(\forall x\, P(x,y))[x/y] = \forall u\, (P(x,y))[x/y, u/x]$
$= \forall u\, P(u,x)$.

2) $(\forall x\, P(x,y))[y/x] \neq \forall y\, P(y,y)$ because the variable x is not free in $\forall x\, P(x,y)$

Instead: $(\forall x\, P(x,y))[y/x] = \forall x\, P(x,y)$. (The substitution ignores x.)

Substitution lemma. Let $x_1, \dots, x_n$ be distinct variables and $t_1, \dots, t_n$ terms.

For any formula φ and any interpretation $\mathcal{M}, v$,

$$\mathcal{M}, v \models \varphi[{}^{t_1}/x_1, \dots, {}^{t_n}/x_n] \text{ if, and only if, } \mathcal{M}, v' \models \varphi$$

where v' is the valuation defined by: $v'(x) := \begin{cases} \bar{v}(t_i) & \text{if } x = x_i \text{ for some } 1 \le i \le n \\ v(x) & \text{otherwise} \end{cases}$.

Coincidence lemma. Let φ be a formula, $\mathcal{M}$ a structure, and v_1, v_2 two valuations.

If, for every variable x that occurs freely in φ, we have $v_1(x) = v_2(x)$, then

$$\mathcal{M}, v_1 \models \varphi \text{ if, and only if, } \mathcal{M}, v_2 \models \varphi.$$

Both lemmas are proved by induction on φ. (Exercise.)

$\Gamma \models \varphi$ means: for all $\overbrace{\text{interpretations}}^{:= \text{ a pair with } \mathcal{M} \text{ an } \mathcal{L}\text{-structure}, v \text{ a valuation}}$ $(\mathcal{M}, v)$, if $\underbrace{\mathcal{M}, v \models \Gamma}_{\text{for every } \gamma \in \Gamma,\ \mathcal{M}, v \models \gamma}$ then $\mathcal{M}, v \models \varphi$.

We prove that the rules are sound:

($\forall$E) Suppose $\Gamma \models \forall x \varphi$. Let t be any term. We need to show $\Gamma \models \varphi[t/x]$.

Let $(\mathcal{M}, v)$ be a structure and valuation with $\mathcal{M}, v \models \Gamma$, so $\mathcal{M}, v \models \forall x \varphi$.

Consider v' obtained from v by modifying the value at x: $v'(x) := \bar{v}(t)$.

Since $\mathcal{M}, v \models \forall x \varphi$, we have $\mathcal{M}, v' \models \varphi$.

By the substitution lemma, $\mathcal{M}, v \models \varphi[t/x]$.

($\forall$I) Suppose $\Gamma \models \varphi[y/x]$, where y does not occur freely in $\Gamma \cup \{\forall x \varphi\}$.

We need to show $\Gamma \models \forall x \varphi$. Let $\mathcal{M}, v \models \Gamma$, $a \in M$. We need to show $\mathcal{M}, v[x \mapsto a] \models \varphi$.

Since y is not free in Γ, by the coincidence lemma, $\mathcal{M}, v[y \mapsto a] \models \Gamma$.

Since $\Gamma \models \varphi[y/x]$, we have $\mathcal{M}, v[y \mapsto a] \models \varphi[y/x]$.

Two cases:

- $y = x$: then immediately $\mathcal{M}, v[x \mapsto a] \models \varphi$.
- $y \neq x$: By the substitution lemma, $\mathcal{M}, v[y \mapsto a, x \mapsto a] \models \varphi$.
  Since y is not free in $\forall x \varphi$, y is also not free in φ, since $y \neq x$.
  Thus, by coincidence lemma, $\mathcal{M}, v[x \mapsto a] \models \varphi$.

Soundness of (REFL) is clear, (SUBST) uses the substitution lemma (exercise).

The side condition in $(\forall I)$ is important, otherwise we lose soundness:

Example. (Signature with one unary predicate P)

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{Py \vdash Py}\,(ax)}{Py \vdash \forall x\, Px}\,?}{Py \vdash Pz}\,(\forall E \text{ with } t := z)}{\vdash Py \to Pz}\,(\to I)}{\vdash \forall y \forall z\, (Py \to Pz)}\,(\forall I) \text{ twice}$$

? $\to$ illegal application of $\forall I$, y is free in Py.

But in the structure $\mathcal{M} = (\{0,1\}, P^{\mathcal{M}})$, where $P^{\mathcal{M}} = \{0\}$, we have

$$\mathcal{M} \not\models \forall y \forall z\, (Py \to Pz)$$

since $\mathcal{M}, v \not\models Py \to Pz$, where $v : y \mapsto 0$ and $z \mapsto 1$.

Exercise. Show that the condition "y is not free in $\forall x \varphi$" cannot be omitted from $(\forall I)$.

Zero-o-logical remark. The sentence $\neg \forall x \bot$ is provable in our system:

$$
\dfrac{\dfrac{\dfrac{}{\forall x \bot \vdash \forall x \bot}\ (ax)}{\forall x \bot \vdash \bot}\ (\forall E \text{ with } t := x)}{\vdash \neg \forall x \bot}\ (\to I)
$$

Thus, we must require structures to be non-empty.

Otherwise, if E is a structure with underlying set $\emptyset$, then $E \models \forall x\, \varphi$ for any formula φ.

(because there do not exist any valuations in E).

But then $E \not\models \neg \forall x \bot$, so the semantics where we allow the empty structure is unsound.

We get two dual rules for $\exists$ (recall $\exists x\, \varphi := \neg \forall x \neg \varphi$):

$$\frac{\Gamma \vdash \varphi[t/x]}{\Gamma \vdash \exists x \varphi} \ (\exists I_R)$$

and, if y does not occur freely in any formula in $\Gamma \cup \{\exists x \varphi, \psi\}$:

$$\frac{\Gamma, \varphi[y/x] \vdash \psi}{\Gamma, \exists x \varphi \vdash \psi} \ (\exists I_L)$$

Proof that $(\exists I_R)$ is derivable.

$$
\dfrac{\overset{(hyp)}{\Gamma \vdash \varphi[t/x]} \qquad \dfrac{\dfrac{}{\Gamma, \forall x \neg\varphi \vdash \Gamma, \forall x \neg \varphi}\,(ax)}{\Gamma, \forall x \neg\varphi \vdash \Gamma, \neg\varphi[t/x]}\,(\forall E)}{\dfrac{\Gamma, \forall x \neg\varphi \vdash \bot}{\Gamma \vdash \exists x \varphi}\,(\to I)}\,(\to E)
$$

(recall $\exists x \varphi := \neg \forall x \neg \varphi$.)

Proof that $(\exists I_L)$ is derivable.

$$
\dfrac{\dfrac{\dfrac{\overset{(hyp)}{\Gamma, \varphi[y/x] \vdash \psi}}{\Gamma, \neg\psi \vdash \neg\varphi[y/x]}\,(\text{contraposition admissible})}{\Gamma, \neg\psi \vdash \forall x \neg\varphi}\,(\forall I) \qquad \dfrac{}{\neg\forall x\neg\varphi \vdash \neg\forall x \neg\varphi}\,(ax)}{\dfrac{\Gamma, \neg\forall x \neg\varphi, \neg\psi \vdash \bot}{\Gamma, \neg\forall x \neg\varphi \vdash \psi}\,(C)}\,(\to E)
$$

Exercise. The following rules are also derivable:

$$\frac{\Gamma \vdash \forall x\, \varphi}{\Gamma \vdash \varphi}\ (\forall E'), \qquad \text{if } x \text{ not free in } \Gamma, \qquad \frac{\Gamma \vdash \varphi}{\Gamma \vdash \forall x\, \varphi}\ (\forall I'),$$

$$\frac{\Gamma \vdash \varphi}{\Gamma \vdash \exists x\, \varphi}\ (\exists I_R'), \qquad \text{if } x \text{ not free in } \Gamma \cup \{\psi\}, \qquad \frac{\Gamma, \varphi \vdash \psi}{\Gamma, \exists x\, \varphi \vdash \psi}\ (\exists I_L'),$$

$$\frac{\Gamma \vdash s \doteq t}{\Gamma \vdash t \doteq s}\ (\text{SYMM}) \qquad \text{and} \qquad \frac{\Gamma_1 \vdash s \doteq t \qquad \Gamma_2 \vdash t \doteq u}{\Gamma_1 \cup \Gamma_2 \vdash s \doteq u}\ (\text{TRANS}).$$

Lemma. (Syntactic compactness) If $\Gamma \vdash \varphi$ then there is finite $\Gamma' \subseteq \Gamma$ such that $\Gamma' \vdash \varphi$.

Proof. By induction on the proof of $\Gamma \vdash \varphi$. We just mention two cases (other cases: exercise):

- (ax) We can take $\Gamma' := \{\varphi\}$.
- $(\to I)$ $\dfrac{\Gamma, \varphi \vdash \psi}{\Gamma \vdash \varphi \to \psi}$ By IH, pick $\Gamma' \subseteq \Gamma$ finite with $\Gamma', \varphi \vdash \psi$. Then $\Gamma' \vdash \varphi \to \psi$ by $(\to I)$.

$\square$

Examples of theories.

- For any $n \geq 2$, let $\varphi_n := \exists x_1 \dots \exists x_n \left(\bigwedge_{i=1}^{n-1} \bigwedge_{j=i+1}^{n} \neg (x_i \doteq x_j) \right)$, and $T_\infty := \{ \varphi_n : n \geq 2 \}$.
  Then T_∞ is a theory in the empty signature, so structures are sets, and $\mathcal{M} \models T_\infty$ iff M is infinite.
- Directed graphs, partial and total orders, equivalence relations, groups, rings, fields, PA, ZF, ... (see TD)
- In the signature with a constant 0 and a unary function S, consider the two axioms:

  1) $\forall x \neg Sx \doteq 0$

  2) $\forall x \forall y \, (Sx \doteq Sy \rightarrow x \doteq y)$

  Also consider the monadic second order sentence:

  3) $\forall X \left(\left(X(0) \wedge \forall x (X(x) \rightarrow X(Sx)) \right) \rightarrow \forall y \, X(y) \right)$

  In a structure $\mathcal{M}$, X is interpreted as a subset of the underlying set of $\mathcal{M}$.

Then any structure $\mathcal{M}$ satisfying (1), (2), (3) is isomorphic to $(\mathbb{N}, 0, +1)$. (Dedekind's Theorem)

There does not exist a first order theory that captures exactly $(\mathbb{N}, 0, +1)$ up to isomorphism.

To see why, we need to develop some more methods for analyzing FO logic.

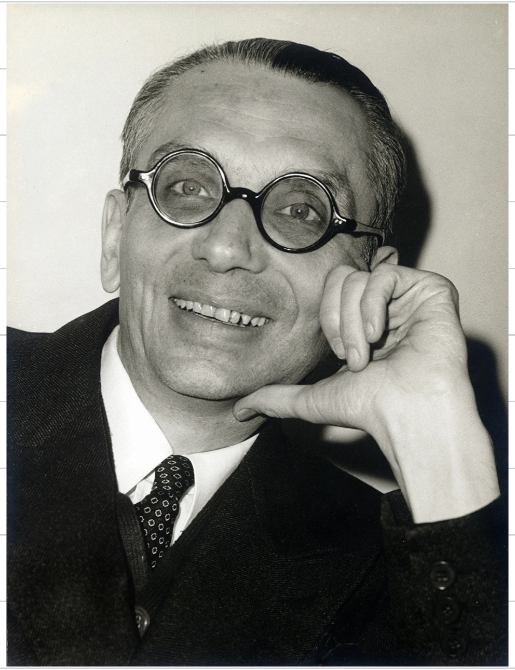
7. Completeness of FO

Gödel's link between syntax and semantics

Theorem (Completeness of FO logic). (Gödel, 1929) For any set Γ of first-order formulas and φ a first-order formula:

if $\Gamma \models \varphi$ then $\Gamma \vdash \varphi$.

We will look at a proof by:



Kurt Gödel (1906-1978)

https://1000wordphilosophy.com/2024/06/01/godel/

Leon Henkin (1921-2006)

https://en.wikipedia.org/wiki/Leon_Henkin

Let Γ be a set of formulas. We say that Γ is consistent if $\Gamma \nvdash \bot$ and satisfiable if $\Gamma \not\models \bot$, i.e., if there exists an interpretation $\mathcal{M}, v$ such that $\mathcal{M}, v \models \Gamma$.

We will prove:

For any set of formulas Γ, if Γ is consistent, then Γ is satisfiable.

This suffices: if $\Gamma \nvdash \varphi$ then $\Gamma \cup \{\neg\varphi\}$ is consistent (using the (C) rule) and therefore $\Gamma \cup \{\neg\varphi\}$ is satisfiable, from which $\Gamma \not\models \varphi$ follows.

Remark. If for all finite subsets Γ' of Γ, Γ' is consistent, then Γ is consistent, by syntactic compactness.

The analogous statement for satisfiable sets is also true, but much harder to prove.

This will be the compactness theorem for FO logic, which we will deduce from completeness.

Let Γ be a consistent set of formulas.

Suppose $\mathcal{M}$ is a structure and $v : V \to M$.

We define $Th(\mathcal{M}, v) := \{\varphi$ a formula such that $\mathcal{M}, v \models \varphi\}$.

We need to show that there exists $(\mathcal{M}, v)$ such that $\Gamma \subseteq Th(\mathcal{M}, v)$.

Plan

1) construct a set T of formulas containing Γ and satisfying some special properties;
2) for a set T with these special properties, construct $\mathcal{M}, v$ such that $T = Th(\mathcal{M}, v)$.

The special properties of the set of formulas T are the following:

1) for any formula φ and variable x, there exists a term t such that

$$T \vdash \exists x \varphi \to \varphi[^t/_x]$$ (witness property);

2) for any formula φ, either $T \vdash \varphi$ or $T \vdash \neg\varphi$ (negation-complete).

Remark. If $T = Th(\mathcal{M}, v)$, then T is negation-complete, and has an extension by constants with the witness property.

Proposition 1. Let T be a consistent and negation complete set with the witness property.
("Henkin's Theorem") There exists an interpretation $(\mathcal{M}, v)$ such that $\mathcal{M}, v \models T$.

We say: $\mathcal{M}, v$ is a **model of** T.

Proposition 2. Let Γ be a consistent set of L-formulas. There exist an extension L' of L by constants and a set T of L'-formulas such that $\Gamma \subseteq T$, T is consistent, negation-complete and has the witness property.

Proofs: On the board.

Together, these two propositions yield Gödel's Theorem:

If Γ is consistent, pick T as in Prop. 2, and $(\mathcal{M}, v)$ as in Prop. 1.

Since $\Gamma \subseteq T$, we in particular have $\mathcal{M}, v \models \Gamma$.

Compactness of FO logic. For any set Γ of formulas, if $\Gamma \models \varphi$, then there exists a finite $\Gamma' \subseteq \Gamma$ such that $\Gamma' \models \varphi$.

Proof. Suppose $\Gamma \models \varphi$.

By completeness, $\Gamma \vdash \varphi$.

There exists $\Gamma' \subseteq \Gamma$ finite such that $\Gamma' \vdash \varphi$ (Syntactic compactness).

By soundness, $\Gamma' \models \varphi$. $\square$

Let $\mathcal{M}, \mathcal{N}$ be L-structures. We write

$$Th(\mathcal{M}) := \{\varphi \text{ an } L\text{-sentence} \mid \mathcal{M} \models \varphi\}$$

and $\mathcal{M}$ is elementarily equivalent to $\mathcal{N}$ ($\mathcal{M} \equiv \mathcal{N}$) if $Th(\mathcal{M}) = Th(\mathcal{N})$.

Note that the definition of $\equiv$ depends on L! We write $\equiv_L$ if we want to emphasize this.

We say $\mathcal{M}'$ is a submodel of $\mathcal{M}$ if $M' \subseteq M$, $R^{\mathcal{M}'} = R^{\mathcal{M}} \cap M'^n$ for all n-ary R, and $f^{\mathcal{M}'} = f^{\mathcal{M}}|_{M'^n}$ for all n-ary f.

A submodel $\mathcal{M}'$ is elementary submodel if $\mathcal{M}' \equiv_{L^{+M'}} \mathcal{M}$ where

$$L^{+M'} := L \sqcup \{c_a \mid a \in M'\}.$$

Löwenheim-Skolem theorems show that one can find models of varying cardinality.

Theorem. (downward L.S.) Any L-structure has an elementary submodel of cardinality at most the set of L-formulas.

In particular, if L is countable, and T is a consistent set of L-sentences, then T has a countable model, by completeness + DLS.

Theorem (∞ L.S.) Let T be a set of L-formulas and suppose that, for all $n \in \mathbb{N}$, there exists a model $\mathcal{M}$ of T with $|\mathcal{M}| \geq n$. Then T has an infinite model.

Theorem. (upward L.S.) If $\mathcal{M}$ is a model of a set T, then for any cardinal $\kappa \geq |\mathcal{M}|$, there exists a model $\mathcal{M}'$ of T with $|\mathcal{M}'| \geq \kappa$.

We omit the proofs, which can be done by compactness.

Compactness can be used to show that certain concepts are not FO-definable.

Example Let $L = \{R^{(2)}\}$, so L-structures are directed graphs.

A set T of L-sentences defines connectedness if, for any L-structure G, we have $G \models T$ iff G is connected.

Claim. There does not exist a set of L-sentences that defines connectedness.

Proof. If T were such a set, then, for any $n \geq 2$, consider

$$\psi_n := \neg(x \doteq y) \wedge \neg \exists x_1 \cdots \exists x_n \left(x \doteq x_1 \wedge y \doteq x_n \wedge \bigwedge_{i=1}^{n-1} R x_i x_{i+1} \right)$$

Now $T' := T \cup \{\psi_n : n \geq 2\}$ is finitely satisfiable.

By compactness, T' is satisfiable.

If G, v is a model of T', then $G, v \models \psi_n$ for all $n \geq 2$, so G is not connected.

But, since $T \subseteq T'$, $G \models T$, which contradicts the assumption on T. $\square$

Example By upward L.S. there exists an uncountable model $\mathcal{N}$ which is elementarily equivalent to $(\mathbb{N}, 0, 1, +, \cdot)$.

We even have:

Skolem's Theorem There exists a countable model $\mathcal{N}$ which is elementarily equivalent to $\mathbb{N}$ but not isomorphic to $\mathbb{N}$.

Proof. Consider $T := \mathrm{Th}(\mathbb{N}) \cup \{ \neg(x \doteq \underline{n}) \mid n \in \mathbb{N} \}$, where $\underline{n}$ denotes the term $1 + 1 + \cdots + 1$ (n times).

By compactness, T has a model.

By downward L-S, this model can be made countable, call it $\mathcal{M}, v$.

But the interpretation $v(x)$ of x in $\mathcal{M}$ prevents an isomorphism $\mathcal{M} \to \mathbb{N}$: any such isomorphism would need to send x to a number $> n$, for all $n \in \mathbb{N}$.

$\mathcal{N}$ is a non-standard model of arithmetic. **Skolem's paradox**: countable model of ZFC.