# Logique – cours 4

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# 7. First-order Completeness

**Gödel's link between syntax and semantics**

# Completeness theorem for first-order logic

**Theorem (Gödel, 1930)**

Let $\Gamma \cup \{\varphi\}$ be a set of formulas of first-order logic.

$$\text{If } \Gamma \models \varphi \text{ then } \Gamma \vdash \varphi \, .$$

# Consistency and satisfiability

Let $\Gamma$ be a set of formulas.

- $\Gamma$ is consistent if $\Gamma \nvdash \bot$, i.e., $\bot$ is not provable from $\Gamma$.
- $\Gamma$ is satisfiable if $\Gamma \not\models \bot$, i.e., there exists a model of $\Gamma$.

An equivalent formulation of the completeness theorem:

**Theorem**

If a set of formulas $\Gamma$ is consistent, then it is satisfiable.

# Henkin's proof of the completeness theorem

Henkin's proof of the completeness theorem has two steps:

1. Extension lemma;
2. Model existence lemma (also called "Henkin's Theorem").

# Extending a consistent set

An extension of a set $\Gamma$ of $L$-formulas is a set $\Delta$ of $L'$-formulas, where $L \subseteq L'$ and $\Gamma \subseteq \Delta$.

## Lemma (Extension Lemma)

Any consistent set $\Gamma$ of $L$-formulas admits an extension $\Delta$ in a signature $L'$ such that:

1. $\Delta$ is witnessing, that is, for any $L'$-formula $\varphi$ and variable $x$, there exists a term $t$ such that

$$\Delta \vdash \exists x \varphi \to \varphi[t/x] \ ,$$

2. $\Delta$ is negation-complete, that is, for any $L'$-formula $\varphi$,

$$\text{either } \Delta \vdash \varphi \text{ or } \Delta \vdash \neg\varphi.$$

# Constructing a model out of syntax

## Lemma (Henkin's Theorem)

A consistent, witnessing, and negation-complete set is satisfiable.

**Proof.**

Let $T$ be a consistent, witnessing, negation-complete set of $L$-formulas. A model $\mathcal{M}_T$ of $T$ is constructed as follows:

- the elements of $\mathcal{M}_T$ are $T$-equivalence classes of terms,
- a variable $x$ is interpreted as $[x]$,
- relation and function symbols are interpreted as:

$$R^{\mathcal{M}}([t_1], \ldots, [t_n]) :\Longleftrightarrow \; T \vdash R(t_1, \ldots, t_n) \; ,$$
$$f^{\mathcal{M}}([t_1], \ldots, [t_n]) := [f(t_1, \ldots, t_n)] \; .$$

For any formula $\varphi$, we have $\mathcal{M}_T \models \varphi$ if, and only if, $T \vdash \varphi$, by induction on $\varphi$, witnessing ($\exists$) and negation-complete ($\neg$). ∎

# Proving the Extension Lemma

## Lemma (Extension Lemma)

Any consistent set $\Gamma$ of $L$-formulas admits a witnessing, negation-complete extension $\Delta$ in a signature $L' \supseteq L$.

In general, proving the Extension Lemma requires a non-constructive choice.

The signature $L'$ will be an extension by constants of $L$.

# Extension by constants

Let $L$ be a signature and let $V$ be a set.

Define the signature

$$L^{+V} := L \sqcup \{c_x \mid x \in V\} ,$$

where each $c_x$ is a constant (i.e., a nullary function).

## Remark

A pair $(\mathcal{M}, v)$ with $\mathcal{M}$ an $L$-structure and $v \colon V \to M$ a valuation

can be identified with

an $L^{+V}$-structure $\mathcal{M}^{+V}$,

by setting $(c_x)^{\mathcal{M}^{+V}} := v(x)$ for each $x \in V$.

# Proof of the Extension Lemma, part 1/2

Proof.

Let $\Gamma$ be a consistent set of $L$-formulas. Define

$$L_0 := L \quad \text{and} \quad \Gamma_0 := \Gamma,$$

and, for any $n \geq 0$, inductively define

$$L_{n+1} := L^{+\mathsf{Form}(L_n)} \quad \text{and}$$
$$\Gamma_{n+1} := \Gamma_n \cup \{\exists x \varphi \to \varphi[c_{\exists x \varphi}/x] \ : \ \exists x \varphi \in \mathsf{Form}(L_n)\} \ ,$$

and define $L' := \bigcup_n L_n$ and $\Gamma' := \bigcup_n \Gamma_n$.

The set $\Gamma'$ is consistent (induction on $n$) and witnessing.

# Proof of the Extension Lemma, part 2/2

Proof (continued).

The following collection of subsets of Form($L'$),

$$\mathcal{P} := \{\Delta \text{ a consistent set of } L'\text{-formulas and } \Gamma' \subseteq \Delta\} \ ,$$

is non-empty and closed under unions of chains.

By Kuratowski-Zorn's Lemma, pick a maximal $\Delta \in \mathcal{P}$.

The set $\Delta$ is negation-complete: if $\Delta \nvdash \neg\varphi$, then $\Delta \cup \{\varphi\}$ is consistent, and thus, by maximality of $\Delta$, we have $\varphi \in \Delta$.

The set $\Delta$ is also

- consistent since $\Delta \in \mathcal{P}$, and
- a witnessing extension of $\Gamma$ since $\Gamma' \subseteq \Delta$. ∎

# Compactness from completeness

**Theorem (Compactness of first-order logic)**

For any set of formulas $\Gamma \cup \{\varphi\}$, if $\Gamma \models \varphi$, then there exists a finite subset $\Gamma'$ of $\Gamma$ such that $\Gamma' \models \varphi$.

*Proof.*

By the completeness theorem, $\Gamma \vdash \varphi$.
Since proofs are finite, pick $\Gamma' \subseteq \Gamma$ finite such that $\Gamma' \vdash \varphi$.
By soundness, $\Gamma' \models \varphi$. ∎

# Satisfiability from finite satisfiability

An immediate application of compactness is the following.

A set of formulas $\Gamma$ is finitely satisfiable if every finite subset of $\Gamma$ is satisfiable.

**Corollary**

If a set of formulas $\Gamma$ is finitely satisfiable, then $\Gamma$ is satisfiable.

**Proof.**

By contraposition. Suppose $\Gamma$ is not satisfiable, i.e., $\Gamma \models \bot$.
By compactness, pick a finite subset $\Gamma'$ of $\Gamma$ such that $\Gamma' \models \bot$. ■

# Prenex normal form

A formula $\varphi$ is in prenex normal form if it has the form

$$Q_1 x_1 \dots Q_n x_n \psi$$

where, for each $1 \leq i \leq n$, $Q_i \in \{\exists, \forall\}$, and $\psi$ is quantifier-free.

## Lemma

For any formula $\varphi$, there exists a formula $\varphi'$ in prenex normal form such that $\vdash \varphi \leftrightarrow \varphi'$.

## Proof (sketch).

By induction, renaming variables and using facts such as

$$\vdash [\psi \rightarrow \forall x \varphi] \leftrightarrow [\forall x (\psi \rightarrow \varphi)]$$

whenever $x$ does not appear freely in $\psi$. ■

# 8. Löwenheim-Skolem theorems

**Changing a model's cardinality without changing its theory**

# First-order logic and cardinality

A limitation of first-order logic is that it cannot count precisely:

We will see that FO cannot distinguish finite from infinite.

The Löwenheim-Skolem theorems will show that FO also cannot distinguish infinite cardinalities from each other either.

# A set of formulas defining infinity

For every $n \in \mathbb{N}_{\geq 1}$, define the sentence (in the empty signature):

$$\psi_n := \exists x_1 \ldots \exists x_n \left( \bigwedge_{i=1}^{n} \bigwedge_{j=i+1}^{n} x_i \neq x_j \right) .$$

A structure $\mathcal{M}$ satisfies $\psi_n$ if, and only if, $|M| \geq n$.

The set $\Psi := \{\psi_n \ : \ n \in \mathbb{N}_{\geq 1}\}$ does not have any finite models, but any infinite structure satisfies $\Psi$.

In other words, $\Psi$ defines infinity.

# Infinite models from unbounded finite models

## Proposition

Let $\Gamma$ be a set of sentences such that, for every $n \in \mathbb{N}$, there exists a model of $\Gamma$ of cardinality at least $n$. Then $\Gamma$ has an infinite model.

## Proof.

The set $\Delta := \Gamma \cup \Psi$ is finitely satisfiable:

> If $\Delta' \subseteq \Delta$ is finite, let $n$ be maximal such that $\psi_n \in \Delta'$.
>
> Pick a model of $\Gamma$ of cardinality $\geq n$.

By compactness, $\Delta$ is satisfiable. Pick a model $\mathcal{M}$ of $\Delta$.

Then $\mathcal{M} \models \Gamma$, and $M$ is infinite, since $\mathcal{M} \models \Psi$. ∎

# Finiteness is not definable by a first-order sentence

### Proposition

There does not exist a set of first-order sentences $\Gamma$ such that

$$\mathcal{M} \models \Gamma \text{ if, and only if, } M \text{ is finite.}$$

### Proof.

Let $\Gamma$ be a set of first-order sentences valid in all finite models.

Then $\Gamma$ has models of cardinality $n$ for every $n \in \mathbb{N}$.

By the previous proposition, $\Gamma$ has an infinite model. ■

# Elementary equivalence

Let $\mathcal{M}$ be an $L$-structure. We write

$$\mathrm{Th}\,\mathcal{M} := \{\varphi \text{ an } L\text{-sentence } \mid \mathcal{M} \models \varphi\}\ ,$$

and call this the theory of $\mathcal{M}$.

An $L$-structure $\mathcal{M}$ is elementarily equivalent to an $L$-structure $\mathcal{S}$ if

$$\mathrm{Th}\,\mathcal{M} = \mathrm{Th}\,\mathcal{S}\ ,$$

and we write $\mathcal{M} \equiv \mathcal{S}$ in this case. (Also called FO-equivalent.)

Note: the notion of FO-equivalence depends on the signature $L$.
We sometimes write $\mathrm{Th}_L(\mathcal{M})$ and $\mathcal{M} \equiv_L \mathcal{S}$ to avoid ambiguity.

# Submodels

A model $\mathcal{S}$ is a submodel of $\mathcal{M}$ if:

- $S \subseteq M$,
- $R^{\mathcal{S}} = R^{\mathcal{M}} \cap S^n$ for every $n$-ary predicate $R$,
- $f^{\mathcal{S}} = f|_{M^n}$ for every $n$-ary function $f$.

# Elementary submodels

A submodel $\mathcal{S}$ of $\mathcal{M}$ is an elementary submodel if

$$\mathcal{S}^{+S} \equiv_{L^{+S}} \mathcal{M}^{+S} \ ,$$

where, for each $s \in S$, the constant $c_s$ is interpreted as $s$;

in other words:

for any first-order sentence $\varphi$, which can contain elements of $S$,

$$\mathcal{S} \models \varphi \text{ if, and only if, } \mathcal{M} \models \varphi \ .$$

### Theorem (Downward Löwenheim-Skolem Theorem)

Let $\mathcal{M}$ be an $L$-structure. Then $\mathcal{M}$ has an elementary submodel of cardinality at most that of the set of $L$-formulas.

# A proof of the DLS theorem

Proof (Karagila, 2014).

For an $L$-structure $\mathcal{M}$, a subset $A$ of $M$, and a formula $\exists x \varphi(x, \bar{y})$, an $A$-Skolem function for $\varphi$ is a function

$$f\colon \{\bar{a} \in M^{\bar{y}} \ : \ \mathcal{M} \models \exists x \varphi(x, \bar{a})\} \to M$$

such that $\mathcal{M} \models \varphi(f(\bar{a}), \bar{a})$ for each $\bar{a} \in \operatorname{dom} f$.

Let $A_0 := \emptyset$, and let $F_0$ be a set of $\emptyset$-Skolem functions. Inductively define, for any $n \geq 0$,

$$A_{n+1} := A_n \cup \{\operatorname{im} f \mid f \in F_k\} \ ,$$

and choose, for each $f \in F_n$, an extension $f'$ to an $A_{n+1}$-Skolem function. Let $F_{n+1}$ be the resulting set of functions.

The submodel on the set $\bigcup_n A_n$ does the job (exercise). ■

# Skolem's Theorem from DLS

Corollary (Skolem's Theorem, 1919)

Let $\Gamma$ be countable set of sentences.

If $\Gamma$ is satisfiable, then $\Gamma$ has a model of cardinality $\leq \aleph_0$.

Proof of Corollary.

Since $\Gamma$ is countable, we may assume the signature is countable, by restricting to the symbols actually occurring in $\Gamma$.

Pick a model $\mathcal{M}$ of $\Gamma$.

By the DLS theorem, pick an elementary submodel $\mathcal{S}$ of $\mathcal{M}$ of cardinality $\leq \aleph_0$. Then $\mathcal{S}$ is still a model of $\Gamma$. ■

# Making models of larger cardinality

**Theorem (Upward Löwenheim-Skolem Theorem)**

Let $\Gamma$ be a set of formulas with an infinite model.

For any cardinal $\kappa$, the set $\Gamma$ has a model of cardinality at least $\kappa$.

*Proof.*

An exercise in applying compactness. ∎

# Making models of a precise infinite cardinality

**Theorem**

Let $\Gamma$ be a set of formulas with an infinite model.
For any infinite cardinal $\kappa$ with $\kappa \geq |\Gamma|$, there exists a model of $\Gamma$ of cardinality exactly $\kappa$.

*Proof.*

Combine DLS, ULS, and compactness (exercise). ∎

# 9. Models, axiomatizations, and theories

**The meta-theory of first-order logic**

# Elementary axiomatizations

Let $L$ be a signature and let $\mathcal{K}$ be a class of $L$-structures.

An elementary axiomatization of $\mathcal{K}$ is a set $\Gamma$ of sentences such that, for any $L$-structure $\mathcal{M}$, we have

$$\mathcal{M} \in \mathcal{K} \iff \mathcal{M} \models \Gamma \,.$$

The class $\mathcal{K}$ is:

- elementary if it has an elementary axiomatization,
- basic elementary if it has a finite elementary axiomatization.

Warning: In some literature, 'elementary' $\mapsto$ '$\Delta$-elementary' and 'basic elementary' $\mapsto$ 'elementary'.

# Theories

A theory is a set $T$ of $L$-sentences closed under deduction, i.e.,

$$\text{if } T \vdash \varphi \text{ then } \varphi \in T \ .$$

The deductive closure of a set of $L$-sentences $S$ is the smallest theory containing $S$.

The theory of a class $\mathcal{K}$ of $L$-structures is

$$\begin{aligned} \operatorname{Th} \mathcal{K} &:= \bigcap_{\mathcal{M} \in \mathcal{K}} \operatorname{Th} \mathcal{M} \\ &= \{\varphi \text{ an } L\text{-sentence } \mid \mathcal{M} \models \varphi \text{ for every } \mathcal{M} \text{ in } \mathcal{K}\} \ . \end{aligned}$$

**Remark**

For any $\mathcal{K}$, $\operatorname{Th} \mathcal{K}$ is a theory.

# Models

The class of models of a set of sentences $\Gamma$ are the $L$-structures in the class

$$\operatorname{Mod}\Gamma := \{\mathcal{M} \text{ an } L\text{-structure } \mid \mathcal{M} \models \Gamma\} .$$

Warning. If $\operatorname{Mod}\Gamma$ is not empty, then it is a proper class, i.e., it is not a set, as it contains structures of arbitrarily large cardinality.

# The model–theory connection

For any set of $L$-sentences $\Gamma$ and any class of $L$-structures $\mathcal{K}$:

$$\Gamma \subseteq \mathrm{Th}\,\mathcal{K} \iff \mathcal{K} \subseteq \mathrm{Mod}\,\Gamma \ ,$$

a Galois connection.

In particular, $T \subseteq \mathrm{Th}\,\mathrm{Mod}\,T$ and $\mathcal{K} \subseteq \mathrm{Mod}\,\mathrm{Th}\,\mathcal{K}$.

**Remark**

A class $\mathcal{K}$ is elementary if, and only if, $\mathrm{Mod}\,\mathrm{Th}\,\mathcal{K} \subseteq \mathcal{K}$.

**Remark**

For any set of sentences $\Gamma$, $\mathrm{Th}\,\mathrm{Mod}\,\Gamma$ is the deductive closure of $\Gamma$.

# Examples of (non-)elementary classes

We gave an elementary axiomatization $\Psi$ of the class of infinite $L$-structures. (The same axiomatization works for any $L$.)

Many classes of structures in mathematics are basic elementary: groups, rings, fields, graphs, ....

We also showed that the class of finite structures is not elementary.

Similar arguments can be used to show that, for example, the class of connected graphs is not elementary.

# Numerals

Consider the signature with a constant $\mathbf{0}$ and a unary function $(-)'$ (written in postfix notation). The structure $\mathcal{N}$ has underlying set $\mathbb{N}$, interpreting $\mathbf{0}$ as $0$ and $n'$ as $n+1$ for each $n \in \mathbb{N}$.

For any $n \in \mathbb{N}_{\geq 1}$, we define a term $\mathbf{n}$, by induction on $n$:

$$\text{if } n = 0 \text{, then } \mathbf{n} := \mathbf{0}, \text{ and if } n = s+1 \text{, then } \mathbf{n} := \mathbf{s}' \; .$$

In other words, $\mathbf{n}$ is the term $\mathbf{0}$ followed by $n$ applications of $(-)'$.

We call $\mathbf{n}$ the numeral of the number $n$.

**Remark**

In the structure $\mathcal{N}$, the interpretation of $\mathbf{n}$ is $n$:

$$\mathbf{n}^{\mathcal{N}} = n \; .$$

# A non-standard model

### Theorem (Skolem)

There is a countable model of $\mathrm{Th}\,\mathcal{N}$ that is not isomorphic to $\mathcal{N}$.

### Proof.

Let $L' := \{\mathbf{0}, (-)', c\}$ with $c$ a constant, and consider

$$\Gamma := \mathrm{Th}(\mathcal{N}) \cup \{\neg(c = \mathbf{n}) \mid n \in \mathbb{N}\} \ .$$

The set $\Gamma$ is finitely satisfiable (in $\mathcal{N}$), hence $\Gamma$ is satisfiable in an at most countable model, by DLS.

Since $\mathrm{Th}\,\mathcal{N}$ does not have finite models, $\Gamma$ must have a countable model $\mathcal{M}$. We show that $\mathcal{M}$ is not isomorphic to $\mathcal{N}$:

> For any homomorphism $f : \mathcal{N} \to \mathcal{M}$, we have $f(n) = \mathbf{n}^{\mathcal{M}}$ for every $n \in \mathbb{N}$. Thus, $c^{\mathcal{M}}$ is not in the image of $f$. ■

# Independent sentences and incomplete theories

Let $T$ be a theory in a signature $L$, and let $\varphi$ be an $L$-sentence.

- $T$ proves $\varphi$, notation $\vdash_T \varphi$, if $\varphi \in T$, i.e., $T \vdash \varphi$.
- $T$ refutes $\varphi$, notation $\vdash_T \neg\varphi$, if $\neg\varphi \in T$, i.e., $T \vdash \neg\varphi$.
- $T$ determines $\varphi$ if $T$ either proves or refutes $\varphi$.
- $\varphi$ is independent from $T$ if $T$ does not determine $\varphi$.

The theory $T$ is complete if, for every sentence $\varphi$, $T$ determines $\varphi$.

The theory $T$ is incomplete if it is not complete, i.e., if there exists a sentence that is independent from $\varphi$.

Note. Some authors use 'decides' instead of 'determines' and 'undecidable sentence' instead of 'independent' sentence.

# An incomplete theory

For any $L$-structure $\mathcal{M}$, the theory $\mathrm{Th}(\mathcal{M})$ is complete.

However, if $\mathcal{K}$ is a *class* of $L$-structures, $\mathrm{Th}(\mathcal{K})$ need not be complete.

### Example

Let $L = \{\cdot\}$ and let $\mathcal{K}$ be the class of semigroups.
The theory $T := \mathrm{Th}(\mathcal{K})$ is not complete: for instance, the sentence

$$\forall x \forall y (x \cdot y = y \cdot x)$$

is independent from $T$.
A finite elementary axiomatization of $T$ is

$$\{\forall x \forall y \forall z (x \cdot y) \cdot z = x \cdot (y \cdot z)\} \ .$$

# Enumerating valid formulas

Theorem

Let $L$ be a countable signature and $V$ a countable set of variables.
Let $S$ be a decidable set of $L$-sentences.
The set $\{\varphi \text{ an } L\text{-sentence } \mid S \vdash \varphi\}$ is recursively enumerable.

Note. We did not yet properly define what we mean by 'recursively enumerable' or 'decidable' set of formulas. We will do so in the next lecture.

# Complete recursively axiomatized theories are decidable

### Corollary

Let $T$ be a theory which is the deductive closure of a decidable set $S$. Then $T$ is recursively enumerable.

### Theorem

If a theory $T$ is consistent, recursively enumerable and negation-complete, then $T$ is decidable.

### Algorithm.

Let $\varphi$ be an input sentence. We will decide whether or not $\varphi \in T$.
Begin enumerating the sentences in $T$ as $\varphi_0, \varphi_1, \ldots$.
For each $n$, if $\varphi_n = \varphi$, output 'yes', if $\varphi_n = \neg\varphi$, output 'no'. ■

This algorithm terminates because $T$ is negation-complete, and is correct because $T$ is consistent.