# Logique – cours 6

DER d'informatique

ENS Paris-Saclay

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# 11. Incompleteness

**Computably characterizing arithmetic is impossible**

# Recall: Gödel numbering

We have defined a computable injective function $\gamma\colon \mathrm{Form}(L, V) \to \mathbb{N}$, whose image is a recursive set.

Let $S \subseteq \mathrm{Form}(L, V)$. Then $S$ is recursive if $\gamma[S] \subseteq \mathbb{N}$ is recursive, and recursively enumerable if $\gamma[S] \subseteq \mathbb{N}$ is recursively enumerable.

# Recall: Representation of functions and relations

Let $T$ be a theory whose signature contains $\mathbf{0}$ and $'$, and $n \geq 0$.

Let $f\colon \mathbb{N}^n \to \mathbb{N}$ be a function. A formula $\varphi(x_1, \ldots, x_n, y)$ represents $f$ in $T$ if, for any $(p_1, \ldots, p_n) \in \mathbb{N}^n$,

$$\text{if } f(p_1, \ldots, p_n) = q, \text{ then } \vdash_T \forall y(\varphi(\mathbf{p_1}, \ldots, \mathbf{p_n}, y) \leftrightarrow y = \mathbf{q}).$$

Let $R \subseteq \mathbb{N}^n$. A formula $\varphi(x_1, \ldots, x_n)$ represents $R$ in $T$ if, for any $(p_1, \ldots, p_n) \in \mathbb{N}^n$,

$$\text{if } (p_1, \ldots, p_n) \in R \text{ then } \vdash_T \varphi(\mathbf{p_1}, \ldots, \mathbf{p_n}), \text{ and}$$
$$\text{if } (p_1, \ldots, p_n) \notin R \text{ then } \vdash_T \neg\varphi(\mathbf{p_1}, \ldots, \mathbf{p_n}).$$

# Diagonalization

For any formula $\varphi$, define $\ulcorner\varphi\urcorner$ to be the numeral of $\gamma(\varphi)$.

The diagonalization of $\varphi$ is the formula $\Delta_\varphi$ defined as

$$\exists x(x = \ulcorner\varphi\urcorner \wedge \varphi) \ .$$

For any $n \in \mathbb{N}$, define $\delta(n) := \begin{cases} \gamma(\Delta_\varphi) & \text{if } n = \gamma(\varphi), \\ 0 & \text{otherwise.} \end{cases}$

## Lemma

The function $\delta$ is recursive.

## Lemma (Diagonal Lemma)

Let $T$ be a theory in which $\delta$ is representable. For any formula $\varphi(y)$, there exists a sentence $G$ such that

$$\vdash_T G \leftrightarrow \varphi(\ulcorner G \urcorner) \ .$$

Proof.

Let $D(x, y)$ be a formula representing $\delta$.
Define $F := \exists y(D(x, y) \land \varphi(y))$, and $n := \gamma(F)$.
Define $G := \Delta_F$, i.e., $G = \exists x(x = \mathbf{n} \land F)$.
Then $\vdash G \leftrightarrow \exists y(D(\mathbf{n}, y) \land \varphi(y))$. $(\star)$
Write $k := \gamma(G)$. Then $\delta(n) = \gamma(\Delta_F) = k$.
Since $D$ represents $\delta$, $\vdash_T \forall y(D(\mathbf{n}, y) \leftrightarrow y = \mathbf{k})$.
Thus by $(\star)$, $\vdash_T G \leftrightarrow \exists y(y = \mathbf{k} \land \varphi(y))$, so $\vdash_T G \leftrightarrow \varphi(\mathbf{k})$.
Since $\ulcorner G \urcorner = \mathbf{k}$, the lemma is proved. ■

## Lemma (Non-self-representability)

Let $T$ be an extension of Q such that $\gamma[T]$ is representable in $T$. Then $T$ is inconsistent.

## Proof.

Since $\delta$ is recursive, $\delta$ is representable in Q, and thus also in $T$. Suppose the formula $\varphi(y)$ represents $\gamma[T]$ in $T$. By the diagonal lemma, pick a sentence $G$ such that

$$\vdash_T G \leftrightarrow \neg\varphi(\ulcorner G \urcorner) .$$

Let $k := \gamma(G)$, so that $\vdash_T G \leftrightarrow \neg\varphi(\mathbf{k})$ $(\star)$. We **claim** that $\vdash_T G$.

Towards a contradiction, suppose $\nvdash_T G$. Then $k \notin \gamma[T]$ as $\gamma$ is injective. Since $\varphi$ represents $\gamma[T]$, $\vdash_T \neg\varphi(\mathbf{k})$. By $(\star)$, $\vdash_T G$, contradiction.

Since $\vdash_T G$, we have $k \in \gamma[T]$. As $\varphi$ represents $\gamma[T]$, $\vdash_T \varphi(\mathbf{k})$. By $(\star)$, we get $\vdash_T \neg G$. So $\vdash_T \bot$. ∎

# Undecidability of arithmetic theories

**Theorem**

There does not exist a consistent, decidable extension of Q.

*Proof.*

Let $T$ be a decidable theory extending Q. By definition, $\gamma[T]$ is a recursive set of numbers. Since decidable sets are representable in Q, in particular $\gamma[T]$ is representable in Q, and thus in $T$.
By the non-self-representability lemma, $T$ is inconsistent. ∎

**Corollary**

The theories Q, PA, and $\mathrm{Th}(\mathcal{N})$ are undecidable.

# Gödel I

**Theorem (Gödel's First Incompleteness Theorem)**

Let $T$ be a consistent recursively enumerable theory which contains Robinson arithmetic Q.
Then there exists a sentence $\varphi$ which is independent from $T$.

*Proof.*

If $T$ would be complete, then $T$ would have to be decidable.
This is impossible, as all consistent extensions of Q are undecidable. ∎

**Corollary**

There does not exist a recursive axiomatization of $\mathrm{Th}(\mathcal{N})$.

# Craig's trick: r.e. theories are recursively axiomatizable

## Proposition (Craig)

Any recursively enumerable theory $T$ is recursively axiomatizable.

## Proof.

Pick a recursive relation $R_T \subseteq \mathbb{N}^2$ such that

$$\gamma[T] = \{x \mid \text{ there exists } y \in \mathbb{N} \text{ such that } (x, y) \in R_T\} .$$

For every sentence $\varphi$ and $n \geq 0$, define $\varphi^{(n)}$ inductively by:

$$\varphi^{(0)} := \varphi, \quad \varphi^{(n+1)} := \varphi \wedge \varphi^{(n)} .$$

The set

$$U := \{\varphi^{(n)} \mid (\gamma(\varphi), n) \in R_T\}$$

is a recursive axiomatization of $T$. ∎

Our proof of Gödel I did not construct an independent sentence.

# Gödel numbering for proofs

Proofs, just like formulas, are finite objects.

As a consequence, Gödel numbering extends to proofs:

- Given a (finite) proof tree, choose a linear ordering on nodes.
- Write the sequents according to the linear order, and write the child relation of the tree as a set of pairs of node addresses.
- View this as a string (over a slightly larger alphabet), and apply Gödel numbering.

We therefore now assume that $\gamma$ can also be applied to proofs.

# Proof and refutation predicates

Let $T$ be a recursively axiomatizable extension of Q.

Define the relations $P_T, R_T \subseteq \mathbb{N}^2$ by:

$$pP_T n :\Longleftrightarrow \text{ there exists a proof } D \text{ of a sentence } \varphi \text{ such that } p = \gamma(D) \text{ and } n = \gamma(\varphi).$$

$$pR_T n :\Longleftrightarrow \text{ there exists a proof } D \text{ of a sentence } \neg\varphi \text{ such that } p = \gamma(D) \text{ and } n = \gamma(\varphi).$$

Exercise. The relations $P_T$ and $R_T$ are recursive.

# Constructing undecidable sentences

Since $P_T$ and $R_T$ are recursive, pick formulas $\pi(x, y)$ and $\rho(x, y)$ which represent the relations in $T$.

Consider the following formula $\varphi(y)$:

$$\forall x(\pi(x, y) \rightarrow \exists z(z < x \wedge \rho(z, y))) \ .$$

By the diagonal lemma, pick a sentence $G$ such that $\vdash_{\mathsf{Q}} G \leftrightarrow \varphi(\ulcorner G \urcorner)$.

**Theorem**

If $T$ is consistent, then the sentence $G$ is independent from $T$.

# Proof of independence of $G$

Suppose that $\vdash_T G$. Pick a proof $D$ of $G$ in $T$.

Write $p := \gamma(D)$ and $n := \gamma(G)$.

Since $pP_T n$ and $\pi$ represents $P_T$, $\vdash_T \pi(\mathbf{p}, \mathbf{n})$.

Since $T$ is consistent, we must have $\nvdash_T \neg G$.

In particular, for every $k < p$, it is not the case that $kR_T n$.

Since $\rho$ represents $R_T$, $\vdash_T \neg\rho(\mathbf{k}, \mathbf{n})$.

However, $\vdash_{\mathsf{Q}} \forall z(z < \mathbf{p} \to \bigvee_{r=0}^{p-1}(z = \mathbf{r}))$. (Exercise.)

It now follows from logical reasoning that $\vdash_T \neg\varphi(\mathbf{n})$.

Since $\vdash_{\mathsf{Q}} G \leftrightarrow \varphi(\ulcorner G \urcorner)$ and $n = \gamma(G)$, we get $\vdash_T \neg G$.

Thus, $\vdash_T G$ and $\vdash_T \neg G$, so $T$ is inconsistent.

The proof that $\nvdash_T \neg G$ is similar. ■

# Gödel vs. Rosser

Gödel actually only proved the first incompleteness theorem under the stronger assumption that $T$ is $\omega$-consistent: for any formula $\varphi(x)$, if $\vdash_T \exists x \varphi(x)$, then there exists $k \in \mathbb{N}$ such that $\nvdash_T \neg\varphi(\mathbf{k})$.

Let $G$ be a sentence such that $\vdash_T G \leftrightarrow \neg\exists x \pi(x, \ulcorner G \urcorner)$.

Exercise. If $T$ is $\omega$-consistent, then $G$ is independent from $T$.

The strengthening we gave above is due to Rosser (1936).

# Provability predicates

We now consider the theory PA, and denote by $\pi(x, y)$ a formula representing the provability relation $P_{\mathsf{PA}}$.

Write $\mathrm{Pr}(y)$ for the formula $\exists x \pi(x, y)$, and, for any formula $\varphi$, write $\Box\varphi$ for the formula $\mathrm{Pr}(\ulcorner\varphi\urcorner)$.

### Theorem (Gödel, Hilbert, Bernays, Löb)

The following three properties hold for any sentences $\varphi, \psi$:

1. if $\vdash_{\mathsf{PA}} \varphi$ then $\vdash_{\mathsf{PA}} \Box\varphi$;
2. $\vdash_{\mathsf{PA}} \Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$;
3. $\vdash_{\mathsf{PA}} \Box\varphi \to \Box\Box\varphi$.

### Lemma (Key Lemma, difficult)

For a recursive unary function $f$, $\vdash_{\mathsf{PA}} (f(x) = 0) \to \Box(f(x) = 0)$.

# Löb's Theorem

## Theorem

For any sentence $\varphi$, if $\vdash_{\mathsf{PA}} \Box\varphi \rightarrow \varphi$, then $\vdash_{\mathsf{PA}} \varphi$.

## Proof.

Let $D(y)$ be the formula $\mathrm{Pr}(y) \rightarrow \varphi$, and $C$ a diagonal sentence for $D$.

1. $\vdash_{\mathsf{PA}} \Box\varphi \rightarrow \varphi$ (assumption)
2. $\vdash_{\mathsf{PA}} C \leftrightarrow D(\ulcorner C \urcorner)$ (definition of diagonal sentence)
3. $\vdash_{\mathsf{PA}} \Box(C \rightarrow D(\ulcorner C \urcorner))$ (first property of $\Box$)
4. $\vdash_{\mathsf{PA}} \Box C \rightarrow \Box D(\ulcorner C \urcorner)$ (second property of $\Box$ applied to 3)
5. $\vdash_{\mathsf{PA}} \Box C \rightarrow (\Box\Box C \rightarrow \Box\varphi)$ (second property of $\Box$ and 4)
6. $\vdash_{\mathsf{PA}} \Box C \rightarrow \Box\varphi$ (third property of $\Box$)
7. $\vdash_{\mathsf{PA}} \Box C \rightarrow \varphi$ (by 1 and 6)
8. $\vdash_{\mathsf{PA}} C$ (by 7 and definition of $D$)
9. $\vdash_{\mathsf{PA}} \Box C$ (by 1), so $\vdash_{\mathsf{PA}} \varphi$ (by 7). ∎

# Gödel II

Theorem (Gödel's Second Incompleteness Theorem for PA)

(If PA is consistent, then)

$$\mathsf{PA} \nvdash \neg \mathsf{Pr}(\ulcorner \bot \urcorner) \; .$$

Proof.

If $\mathsf{PA} \vdash \neg \mathsf{Pr}(\ulcorner \bot \urcorner)$, this means $\mathsf{PA} \vdash \Box\bot \to \bot$.

By Löb's theorem, $\mathsf{PA} \vdash \bot$. ■

"Do Gödel's incompleteness theorems matter?", L. Paulson (2021).

# Excursion: Ramsey theory

A different kind of independent sentence arises in Ramsey theory.

For $r \in \mathbb{N}_{\geq 1}$ and $Y$ a finite set, write $\binom{Y}{r}$ for the set of subsets of $Y$ of cardinality $r$. For $m, s \in \mathbb{N}_{\geq 1}$, an $s$-coloring of $\binom{[m]}{r}$ is a function $\binom{[m]}{r} \to [s]$. The value at $X \in \binom{[m]}{r}$ is the color of $X$.

A subset $Y \subseteq [m]$ is monochromatic, for a given $s$-coloring, if every $X \in \binom{Y}{r}$ has the same color.

**Theorem (Ramsey)**

For any $r, s, n \geq 1$, there exists $m \geq n$ such that any $s$-coloring of $\binom{[m]}{r}$ has a monochromatic set of size at least $n$.

Exercise. For $r = 2$, $s = 2$, $n = 3$, the choice $m := 6$ works.
"If at least six people attend a party, then there are at least three mutual friends or three mutual enemies."

# Paris-Harrington theorem

Let $S \subseteq \mathbb{N}$ be finite and non-empty. We say that $S$ is large if $|S| \geq \max S$. E.g. $\{2, 3, 4\}$ is large, $\{1, 100\}$ is not.

## Proposition

For any $r, s \geq 1$, $n \geq r$, there exists $m \geq n$ such that any $s$-coloring of $\binom{[m]}{r}$ has a *large* monochromatic subset of $[m]$.

Let PH denote a formalization of the statement of this proposition in the language of arithmetic.

## Theorem (Paris and Harrington, 1977)

PH is independent from Peano arithmetic.

The proof shows that $\vdash_{\mathrm{PA}} \mathrm{PH} \to \neg\,\mathrm{Pr}(\ulcorner \bot \urcorner)$, and applies Gödel II.

# Incompleteness: summary and outlook

We've seen:

- Gödel I: Any consistent r.e. theory of arithmetic is incomplete.
  - We first saw an indirect, computability-theoretic proof that independent sentences must exist;
  - and then a proof of Rosser (based on Gödel's original construction) of an independent sentence.
- Gödel II: Peano arithmetic cannot prove its own consistency.

Two questions:

1. What is the logical meaning of 'construction'?
2. What first-order theories are still decidable?

# **12.** Sequent calculus

**Revisiting proof theory**

# Gentzen sequent calculus

Gentzen (1934/35) introduced sequent calculi **LK** and **LJ** when analyzing natural deduction.

A formula is built as before from a signature $L$ and a set of variables $V$, but now $\neg, \wedge, \vee, \rightarrow, \forall, \exists$ are the basic connectives.

A sequent is a pair $(\Gamma, \Delta)$ where $\Gamma, \Delta$ are finite lists of formulas.

Idea: $\Gamma \vdash \Delta$ means
'if all formulas in $\Gamma$, then at least one of the formulas in $\Delta$'.

An single-conclusion sequent is a sequent $(\Gamma, \Delta)$ where $|\Delta| \leq 1$.

# Gentzen's rules

There are three kinds of rules in **LK**:

- Structural rules (6)
- Identity rules (2)
- Logical rules (14)

# Structural rules

Exchange:

$$\frac{\Gamma, \varphi, \psi, \Gamma' \vdash \Delta}{\Gamma, \psi, \varphi, \Gamma' \vdash \Delta}\,\mathcal{L}\mathsf{X} \qquad \frac{\Gamma \vdash \Delta, \varphi, \psi, \Delta'}{\Gamma \vdash \Delta, \psi, \varphi, \Delta'}\,\mathcal{R}\mathsf{X}$$

Weakening:

$$\frac{\Gamma \vdash \Delta}{\Gamma, \varphi \vdash \Delta}\,\mathcal{L}\mathsf{W} \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta, \varphi}\,\mathcal{R}\mathsf{W}$$

Contraction:

$$\frac{\Gamma, \varphi, \varphi \vdash \Delta}{\Gamma, \varphi \vdash \Delta}\,\mathcal{L}\mathsf{C} \qquad \frac{\Gamma \vdash \varphi, \varphi, \Delta}{\Gamma \vdash \varphi, \Delta}\,\mathcal{R}\mathsf{C}$$

# Identity rules

Axiom:

$$\frac{}{\varphi \vdash \varphi}\,\mathsf{Ax}$$

Cut:

$$\frac{\Gamma \vdash \varphi, \Delta \qquad \Gamma', \varphi \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\,\mathsf{Cut}$$

# Rules for negation

$$\frac{\Gamma \vdash \varphi, \Delta}{\Gamma, \neg\varphi \vdash \Delta}\,\mathcal{L}\neg \qquad \frac{\Gamma, \varphi \vdash \Delta}{\Gamma \vdash \neg\varphi, \Delta}\,\mathcal{R}\neg$$

# Rules for conjunction

Left:

$$\frac{\Gamma, \varphi \vdash \Delta}{\Gamma, \varphi \land \psi \vdash \Delta}\,\mathcal{L}1\land \qquad \frac{\Gamma, \psi \vdash \Delta}{\Gamma, \varphi \land \psi \vdash \Delta}\,\mathcal{L}2\land$$

Right:

$$\frac{\Gamma \vdash \varphi, \Delta \qquad \Gamma' \vdash \psi, \Delta'}{\Gamma, \Gamma' \vdash \varphi \land \psi, \Delta, \Delta'}\,\mathcal{R}\land$$

# Rules for disjunction

Left:

$$\frac{\Gamma, \varphi \vdash \Delta \qquad \Gamma', \psi \vdash \Delta'}{\Gamma, \Gamma', \varphi \vee \psi \vdash \Delta, \Delta'} \mathcal{L}\vee$$

Right:

$$\frac{\Gamma \vdash \varphi, \Delta}{\Gamma \vdash \varphi \vee \psi, \Delta} \mathcal{R}1\vee \qquad \frac{\Gamma \vdash \psi, \Delta}{\Gamma \vdash \varphi \vee \psi, \Delta} \mathcal{R}2\vee$$

# Rules for implication

Left:

$$\frac{\Gamma \vdash \varphi, \Delta \qquad \Gamma', \psi \vdash \Delta'}{\Gamma, \Gamma', \varphi \to \psi \vdash \Delta, \Delta'} \mathcal{L}{\to}$$

Right:

$$\frac{\Gamma, \varphi \vdash \psi, \Delta}{\Gamma \vdash \varphi \to \psi, \Delta} \mathcal{R}{\to}$$

# Rules for quantifiers

Universal quantifier:

$$\frac{\Gamma, \varphi[t/x] \vdash \Delta}{\Gamma, \forall x.\varphi \vdash \Delta}\ \mathcal{L}\forall \qquad \frac{\Gamma \vdash \varphi, \Delta}{\Gamma \vdash \forall x.\varphi, \Delta}\ \mathcal{R}\forall$$

where $\mathcal{R}\forall$ may only be applied if $x$ is not free in $\Gamma, \Delta$.

Existential quantifier:

$$\frac{\Gamma, \varphi \vdash \Delta}{\Gamma, \exists x.\varphi \vdash \Delta}\ \mathcal{L}\exists \qquad \frac{\Gamma \vdash \varphi[t/x], \Delta}{\Gamma \vdash \exists x.\varphi, \Delta}\ \mathcal{R}\exists$$

where $\mathcal{L}\exists$ may only be applied if $x$ is not free in $\Gamma, \Delta$.

# **LK** and **LJ**

A sequent $\Gamma \vdash \Delta$ is **LK**-provable if it is the root label of a proof tree that follows the above rules. We then write $\Gamma \vdash_{\mathbf{LK}} \Delta$.

A single-conclusion sequent $\Gamma \vdash \Delta$ is **LJ**-provable, notation $\Gamma \vdash_{\mathbf{LJ}} \Delta$, if it is the root label of a proof tree that follows the above rules, except that $\mathcal{L}\vee$ is replaced by

$$\frac{\Gamma, \varphi \vdash \Delta \qquad \Gamma', \psi \vdash \Delta}{\Gamma, \Gamma', \varphi \vee \psi \vdash \Delta} \, \mathcal{L}\vee'$$

(The rule $\mathcal{L}\vee$ can yield a multi-conclusion sequent even if the premises are single-conclusion.)

Note. Any **LJ**-provable sequent is also **LK**-provable.

# The law of excluded middle

### Proposition (Law of excluded middle, LEM)

For any formula $\varphi$, the sequent $\vdash \varphi \vee \neg\varphi$ is provable in **LK**.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\varphi \vdash \varphi}\,\text{Ax}}{\vdash \varphi, \neg\varphi}\,\mathcal{R}\neg}{\vdash \varphi \vee \neg\varphi, \neg\varphi}\,\mathcal{R}1\vee}{\vdash \neg\varphi, \varphi \vee \neg\varphi}\,\mathcal{R}X}{\vdash \varphi \vee \neg\varphi, \varphi \vee \neg\varphi}\,\mathcal{R}2\vee}{\vdash \varphi \vee \neg\varphi}\,\mathcal{R}C$$

# **LK**-provability, natural deduction, and validity

We admit the following two facts:

**Proposition (Soundness of LK)**

If $\Gamma \vdash \Delta$ is **LK**-provable, then, for any interpretation $\mathcal{I}$ of $\Gamma \cup \Delta$, if $\mathcal{I} \models \gamma$ for all $\gamma \in \Gamma$, then there exists $\delta \in \Delta$ such that $\mathcal{I} \models \delta$.

**Lemma (Simulation of natural deduction in LK)**

If $\Gamma \Rightarrow \bigvee \Delta$ is derivable in the natural deduction system, then $\Gamma \vdash \Delta$ is **LK**-provable.

**Corollary**

A sequent $\Gamma \vdash \Delta$ is **LK**-provable if, and only if, $\bigwedge \Gamma \to \bigvee \Delta$ is valid.

*Proof of Corollary.*

$\Rightarrow$ follows from soundness. $\Leftarrow$ follows from Gödel's completeness theorem together with the simulation lemma. ∎

# Gentzen's theorem

**Theorem ("Hauptsatz", Gentzen 1934)**

The cut rule can be eliminated in both **LJ** and **LK**, i.e., there is an algorithm which transforms any proof tree $\pi$ of $\Gamma \vdash \Delta$ into a proof tree $\pi'$ of $\Gamma \vdash \Delta$ that does not use (Cut).

*Proof.*

See the course *Logique et informatique* (J. Goubault-Larrecq), or Chapter 13 of *Proofs and Types* (J.-Y. Girard, 1989) ■

# The disjunction property in **LJ**

Proposition (Disjunction property)

For any formulas $\varphi$ and $\psi$, if $\vdash \varphi \vee \psi$ is **LJ**-provable, then either $\vdash \varphi$ is **LJ**-provable or $\vdash \psi$ is **LJ**-provable.

Proof.

Consider a proof tree $\pi$ in **LJ** of $\vdash \varphi \vee \psi$.

By Gentzen's Hauptsatz, assume the Cut rule is not used in $\pi$.

The last rule of $\pi$ has a conclusion where the left is empty and the right is a disjunction.

Inspecting all the non-cut-rules of **LJ**, the last rule in $\pi$ must be either $\mathcal{R}1\vee$ or $\mathcal{R}2\vee$.

In the first case, $\vdash \varphi$ is **LJ**-provable, and in the second, $\vdash \psi$. ∎

# **LJ** does not have excluded middle

Corollary

There exists a sentence $\varphi$ such that $\vdash \varphi \vee \neg\varphi$ is not **LJ**-provable.

*Proof.*

For example, take a signature with one unary predicate symbol and $\varphi := \forall x P(x)$.
Neither $\vdash \varphi$ nor $\vdash \neg\varphi$ is **LJ**-provable: **LJ**-provable sequents are **LK**-provable, and thus valid, but clearly $\not\models \varphi$ and $\not\models \neg\varphi$.
By the Disjunction Property of **LJ** (contraposed), we get that $\vdash \varphi \vee \neg\varphi$ is not **LJ**-provable. ∎

# 13. Constructive logic

**Dropping excluded middle and reductio ad absurdum**

# Context

- 1920s: *Grundlagenkrise der Mathematik*, fundamental disagreement between formalism and constructivism.
- Today: intuitionistic logic as an object of study *per se*, not necessarily implying a philosophical position.
- Strong influence of intuitionism on computer science: 'constructive proof' $\leftrightarrow$ program. (Curry-Howard)
- 'constructive' $\supseteq$ 'intuitionistic' (?)
- For more about *constructive mathematics* today, see:

  "Five stages of accepting constructive mathematics", A. Bauer (2017)

  "Intuitionism: An Inspiration?", W. Veldman (2021).

# Intuitionistic logic

Intuitionistic predicate logic over a signature $L$ is the set

$$\mathsf{IQC} := \{\varphi \text{ a first-order sentence } : \text{ the sequent } \vdash \varphi \text{ is } \mathbf{LJ}\text{-provable}\} .$$

Intuitionistic propositional logic IPC is the restriction where $L$ only contains nullary predicate symbols (called propositions), and we omit quantifiers and the four quantifier rules from **LJ**.

# Reductio ad absurdum is not intuitionistically valid

## Proposition

There exists a sentence $\varphi$ such that $\vdash \neg\neg\varphi$ is **LJ**-provable, but $\vdash \varphi$ is not **LJ**-provable.

## Proof.

In fact, for any sentence $\varphi$, sequent $\vdash \neg\neg(\varphi \vee \neg\varphi)$ is **LJ**-provable:

$$\dfrac{\dfrac{\dfrac{\text{(exercise)}}{\neg(\varphi \vee \neg\varphi) \vdash \neg\varphi \wedge \neg\neg\varphi} \qquad \dfrac{\dfrac{\dfrac{}{\neg\varphi \vdash \neg\varphi}\,\text{Ax}}{\neg\varphi, \neg\neg\varphi \vdash}\,\mathcal{L}\neg}{\neg\varphi \wedge \neg\neg\varphi \vdash}\,\mathcal{L}\wedge}{\neg(\varphi \vee \neg\varphi) \vdash}\,\text{Cut}}{\vdash \neg\neg(\varphi \vee \neg\varphi)}\,\mathcal{R}\neg$$

We saw already that $\vdash \varphi \vee \neg\varphi$ is not always **LJ**-provable. ∎